3.0 - 19th and 20th century elastic arches.

3.1 Elastic arches. Theory of ellipse of elasticity, Culmann 1875.

3.1 General assumptions of linear elastic bodies. Assumptions of the theory of the ellipse of elasticity

1. Linear constitutive equations, linear stress tensor, linear infinitesimal rotation and deformation tensors, linear equilibrium equations, position of forces not affected by movements, linear constraint conditions.

2 plane cantilever beam with geometrical axis coinciding with the plane of bending, linear elastic constitutive laws.

3 plane forces: their resultant is the vector c **c** , it is applied to the end section of the cantilever, **c** is a unit vector , scalar c is the intensity,

4 the end cross-section of the beam is rigid. It is represented in the plane in figure 1 by vector **a.**

Therefore: vector **a** rotates around the point C (Chasles's theorem[1]), center of rotation.

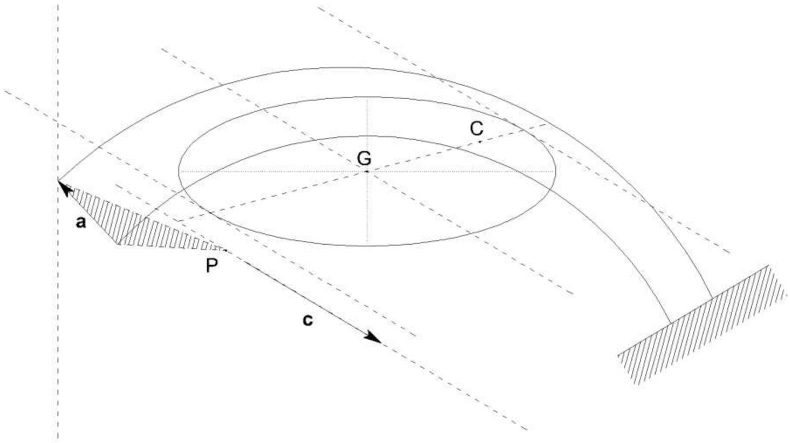


Figure 1 The ellpse of elasticity

Polaity

The correspondence between unit vectors c and points C is one to one. Indeed ,given c theorems on existence and uniqueness of the solution of linear elastic problems require that the point C exists and is unique and vice-versa: the correspondence between the unit vectors c, polars and points C poles is a polarity[2] (figure 1).

Poles and polars are conjugate

If a family of straight lines crosses a centre C, their poles generate the polar **c** of C

1.1. Chasles M.[1] (1793-1880)
1.2. [2]

3-19th and 20th century elastic arches.

Consider the two unit vectors , $\mathbf{c_1}$ e $\mathbf{c_2}$ which have as poles respectively the points C_1 and C_2 (figure 2). By the Theorem of Chasles, we have that the point P, intersection of $\mathbf{c_1}$ and $\mathbf{c_2}$:

$$\mathbf{u_P} = \boldsymbol{\varphi}_1 \times (P - C_1)$$

$$\mathbf{u_P} = \boldsymbol{\varphi}_2 \times (P - C_2)$$

The total motion is then:

$$\mathbf{u_P} = \boldsymbol{\varphi}_1 \times (P - C_1) + \boldsymbol{\varphi}_2 \times (P - C_2)$$

If **k** is the unit vector normal to plane we have :

$$\mathbf{u_P} = \mathbf{k} \times \left[(P - C_1)\varphi_1 + (P - C_2)\varphi_2 \right] = \mathbf{k} \times \sum_i \varphi_i (P - C_i) = \mathbf{k} \times \varphi (P - C)$$

with the assumption:

$$\varphi = \sum_i \varphi_i \ .$$

From the relation you can observe the new centre C is on line $C_1 C_2$.

<u>Law of reciprocity, conjugate elements</u>

Suppose you have on the same cantilever beam two system of forces and movements, the first $\mathbf{c_1}$, C_1, φ_1 and the second c_2 C_2, φ_2 Prove the theorem*: if C_1 belongs to $\mathbf{c_2}$, then C_2 belongs $\mathbf{c_1}$,figure 2.*

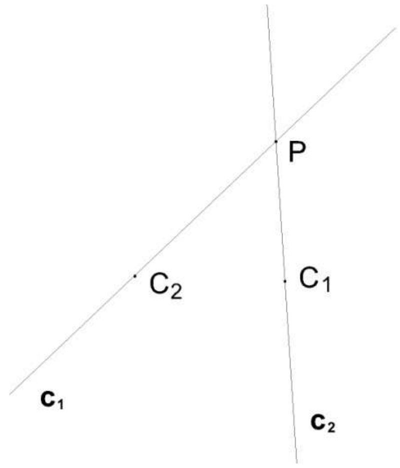


Figure 2

Let's apply Betti 's theorem .
Let it be P the intersection of vectors $\mathbf{c_1}$, $\mathbf{c_2}$

Then the reciprocal work $L_{12} = L_{21}$

$$L_{12} = \boldsymbol{\varphi}_2 \times (P - C_2) \cdot \mathbf{c}_1 = L_{21} = \boldsymbol{\varphi}_1 \times (P - C_1) \cdot \mathbf{c}_2$$

Then:

$$L_{12} = \boldsymbol{\varphi}_2 \cdot (P - C_2) \times \mathbf{c}_1 = L_{21} = \boldsymbol{\varphi}_1 \cdot (P - C_1) \times \mathbf{c}_2$$

But C_1 belongs to $\mathbf{c_2}$, then L_{21} is zero and also L_{12} , then C_2 belongs to $\mathbf{c_1}$.

Two directions such that each contains the pole of the other are known as conjugate .

The polarity has no autoconjugate elements then C cannot belong to **c**

Proof

The work of deformation:

$$L = \frac{1}{2} \mathbf{c} \cdot \mathbf{u}_P = \frac{1}{2} \boldsymbol{\varphi} \cdot (P - C) \times \mathbf{c}$$

If C belongs to **c**, the work is zero .
Then C cannot belong to **c** .

Polarity centre
The polarity centre is the pole of the line at infinite: it is known as G, elastic centre, figure 1.

Ellipse of elasticitry
The polarity centre is the locus of autoconjugate points :then It is an imaginary conic but Its centre G is real.
Let's consider a new polarity product of the old for a symmetry with respect to G .The new correspondence is a polarity, a conic real; the point G, is symmetric of itself, it is the real centre then the conic, without any point at infinite, is an ellipse *the ellipse of elaticity*. With respect to the ellipse, lines **c** and points C correspond each other in a correspondence known as antipolarity. Therefore *the ellipse of elasticity is the conic of the antipolarity between vectors* **c** *and centres* C .

Theorems of Culmann
Let us suppose, that on the geometrical axis of the beam of figure 1 is distributed a system of scalars, the overall amount of which is W, *the elastic weight*, its centre is G. The scalar W is defined later. If the vector **c** passes through G, the antipole is a point at infinity in the direction conjugate, figure 1 . According to Culmann, a couple of moment **m** is equivalent to an infinitesimal force along the line at infinity of the plane . .
Then If the vector **c** is identified with the line at infinity, we obtain the rotation around C=G: .

$$\boldsymbol{\varphi} = W\mathbf{m} \text{ or } \varphi\mathbf{k} = Wm\mathbf{k} \quad 1)$$

then :

$$\varphi = Wm$$

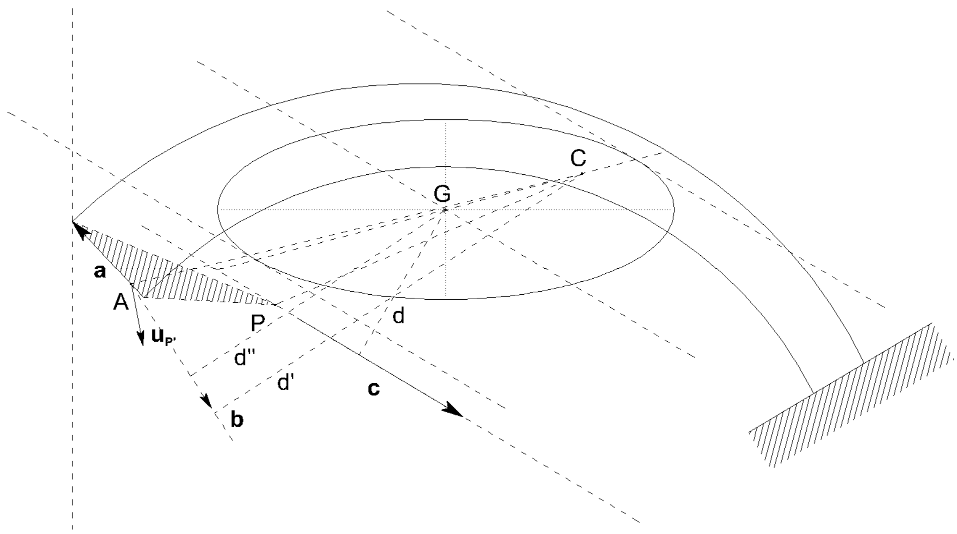


figure 3 *1° theorem The rotation of the unit vector* **a***, caused by a moment* **m** *is equal to the product of* **m** *for the elastic load.*

The displacement of point A, figure 3:

$$\mathbf{u}_A = \boldsymbol{\varphi}\times(A-G) = W\mathbf{m}\times(A-G)$$

Its component along unit vector **b**:

$$W\mathbf{m}\times(A-G)\cdot\mathbf{b} = W\mathbf{m}\cdot(A-G)\times\mathbf{b} = Wmd'' \quad 2)$$

Then:

2° Theorem

the displacement in a direction **b** *of a point* **A** *of the unit vector* **a***, caused by a couple* **m** *is equal to the static moment of the elastic weight with respect to the direction* **b***.*

If the force vector c**c** is in any position and **P** is its point of application ,**a** rotates respect to C, antipole of **c**. We consider its moment with respect to **G**:

$$\mathbf{m} = (P - G) \times c\,\mathbf{c}$$

Therefore **m** e c**c** through G are two vectors equivalent to the vector c**c** for **P**. C is on the straight line CG conjugate of **c,** figure 3. Then the rotation respect to the point **C**is replaced by a translation normal to the conjugate direction GC and the rotation respect to G .

$$\varphi\varphi = W(P - G) \times c\,\mathbf{c} = cWd\mathbf{k} \quad 3)$$

3° theorem

the rotation of the unit vector **a***, caused by a force applied in* P, *c***c,** *is equal to the product of the force* c *for the static moment of elastic weight with respect to the direction of the force itself.*

The displacement of a point A of **a**:

$$\mathbf{u}_A = W(P - G) \times c\,\mathbf{c} \times (A - C) = cWd\mathbf{k} \times (A - C)$$

The component along vector **b ,f**igure 3:

$$\mathbf{u}_A \cdot \mathbf{b} = Wcd\mathbf{k} \times (A - C) \cdot \mathbf{b} = Wcd\mathbf{k} \cdot (A - C) \times \mathbf{b} = cWdd' \quad 4)$$

4° theorem

The displacement of a point A *along direction* **b** *due to a force* **c** *applied in* P *is the product of the magnitude of the force* c *for the centrifugal moment of elastic weight respect to the line of action of the force* **c** *and to line* **b** *.*

Therefore the displacement of the point P, belonging to the force c**c** ,in the direction **c**:

$$cWd^2 \quad 5)$$

5° theorem

The displacement of a point P *belonging to the force in the direction of the force itself is equal to the product of the magnitude of the force for the moment of inertia of the elastic weight with respect to direction considered.*

<u>Basic case. A prismatic cantilever beam</u>

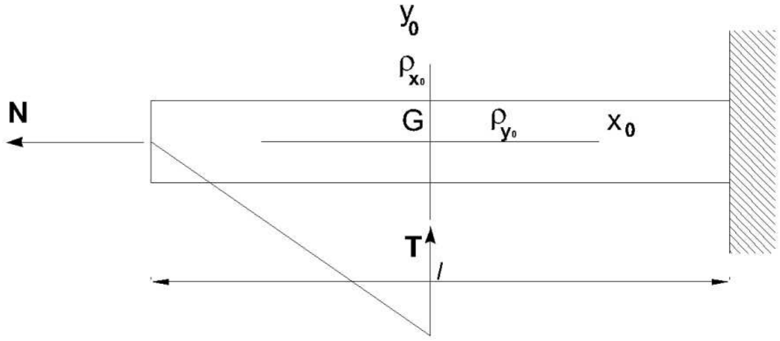


Figure 4 The basic case. A cantilever beam of constant section

3-19th and 20th century elastic arches.

Consider (Figure 4) the case of a cantilever beam of constant section: for symmetry the center of gravity coincide with the geometrical center of the beam and the axes of the ellipse with the axes of symmetry. The case is basic, because it is always possible to reduce to it.
In fact, a couple of moment m applied to the free end deforms the geometric axis of the beam as an arc of a circle of radius $\frac{EJ}{m}$. The rotation of the end section of the beam with respect to the fixed section will be $\frac{ml}{EJ}$ and the center of rotation is the center G of the beam. The weight elastic from theorem 1° is then:

$$W = \frac{l}{EJ} \quad 6)$$

To obtain the axes of ellipse, consider the cases of two forces applied to the end section of the beam . One is a force along the geometric axis, N, and the other a force T through G orthogonal to the geometric axis. The first case generates a translation of the end section of the beam and therefore, applying the theorem, we have:

$$\frac{Nl}{EA} = \frac{Nl}{EJ}\rho_{X_0}{}^2$$

in the formula $\rho_{x_o}^2$ is the square of radius of gyration $\rho_{x_o}^2$

$$\rho_{X_0}{}^2 = \frac{J}{A} \quad 7)$$

The second semi-axis of the ellipse of elasticity is obtained by the force T through G , applied to the end section:it causes the vertical displacement of end section of the beam (disregarding the shear deformation) :

$$\frac{Tl^3}{3EJ} - \frac{Tl^3}{4EJ} = \frac{Tl^3}{12EJ}$$

Then :

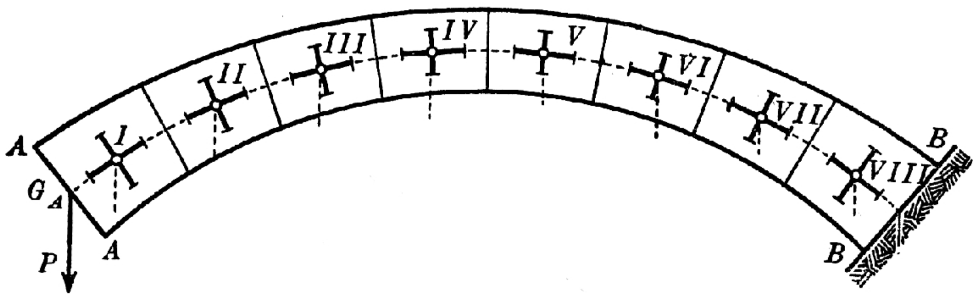
$$\frac{Tl^3}{12EJ} = TW\rho_{y_0}{}^2$$

and

$$\rho_{y_0}{}^2 = \frac{l^2}{12} \quad 8)$$

<u>Arcnes</u>

The process can be extended to plane arches: we divide the arch in prismatic segments. figure 5, and assess for each the ellipse of elasticity. Suppose that all segments are subsequently elastic, one at a time: the principle of superposition allows to evaluate displacements by summations[3]..



1.3. [3] Colonnetti *Scienza delle costruzioni* Einaudi.Torino

Figure 5. The arch is divided in prismatic elements, each effect added according to the assumptions **3.1**., and the principle of superposition

It should be noted that the ellipse of elasticity allows a significant simplification in the calculus of statically indeterminate unknowns.
Consider a statically indeterminate arch, for example the doubly fixed symmetric arch, in the example of figure 6, loaded by a point force **p**. It is well known that this structure is three times indeterminate. Proceeding with the method of the equations of congruence, you choose, for example, the fixed beam shown in figure 6. Usually the parameters of the reaction $\mathbf{r_A}$ are the unknowns $\mathbf{H_A}$, $\mathbf{v_A}$, $\mathbf{m_A}$. Let us choose, instead, as parameters, a system of equivalent forces through G with reference to axes x_0 y_0 of the ellipse of elasticity: their parametres are $\mathbf{H_A}$, $\mathbf{v_A}$, $\mathbf{m_G}$. This choice greatly simplifies the calculus of unknowns. Indeed the congruence equations are the following :

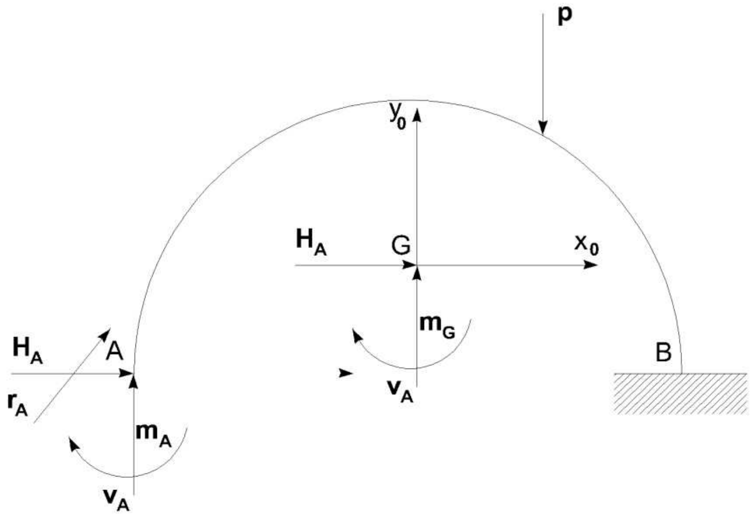


Figure 6

$$\eta_{i0} + \eta_{ij} x_j = 0 \qquad 9)$$

x_i are the indeterminate unknowns, η_{i0} the displacement along x_i if x_j =0,**p**$\neq$ 0, η_{ij} the displacement along x_i if x_j=1 **, p**=0. In the example the unknowns x_i are, one by one,
$\mathbf{H_A}$ $\mathbf{v_A}$ e $\mathbf{m_G}$ .For Maxwlell's theorem and for the properties of the ellipse of elasticity, $\eta_{ij} = 0$ per $i \neq j$ then we obtain three equations in three unknowns, each uncoupled from the others:

$$\begin{aligned} m_G W - pS_p &= 0 \\ v_A J_{y_0} - pJ_{py_0} &= 0 \\ H_A J_{x_0} - pJ_{px_0} &= 0 \end{aligned} \qquad 10)$$

If you have an example with many point forces, it can be traced back to the former with the principle of superposition..

3.2 Casus studii. A reinforced concrete arch bridge of the mid-twentieth century.

3-19th and 20th century elastic arches.

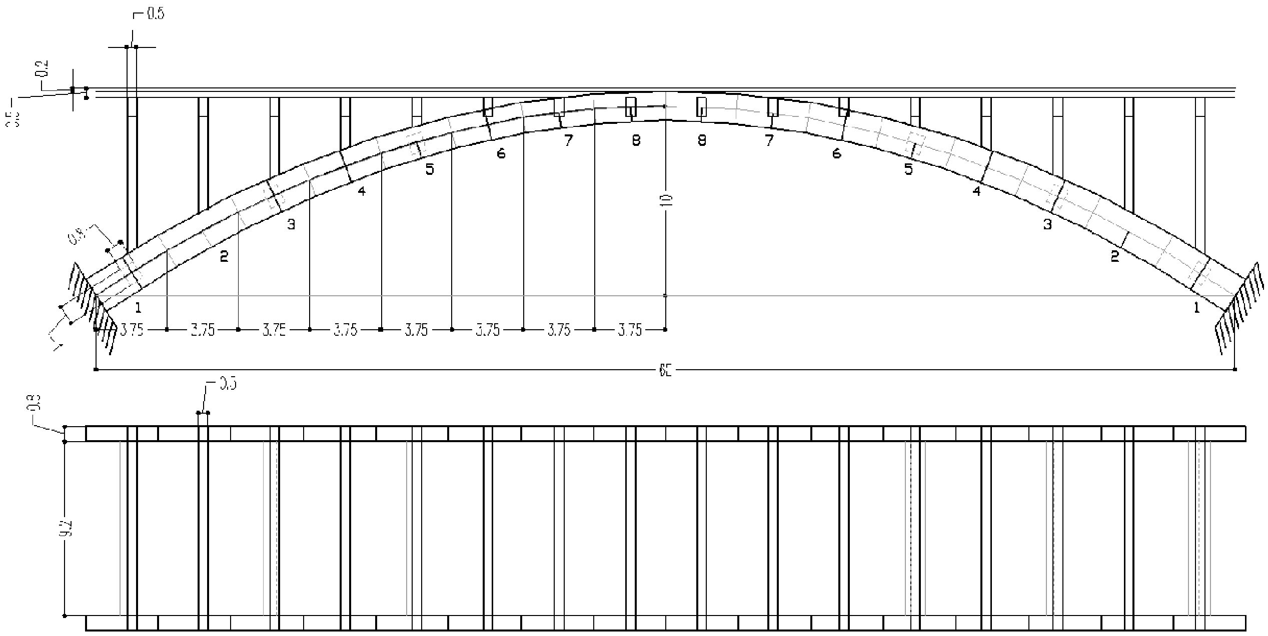


Figure 1 Plan and section of the reinforced concrete of the mid 20th century bridge.

The figure1 shows the vertical section and the horizontal plan of a reinforced bridge built in the mid-twentieth century near Marzabotto (Bologna). Structural analysis is performed with the theory of the ellipse of elasticity..
The principal beams of the bridge are made with two symmetrical parabolas, span 60 m, rise 10 m .
Eeach parabola axis is referred to a Cartesian coordinate system x,y, not represented in the figure whose origin (0,0) is the left support , The symmetric parabola crosses the key (30.00,10.00) and the right support (60.00,0.00).
Each arch was divided into 16 elastic segments as in figure 5 .

The main dimensions of the bridge can be obtained from Figure 1 and from Table 3.
In Table 1, the coordinates of the centerline at the ends of each segments are shown..

X	Y
0	0
3,75	2,34
7,50	4,38
11,25	6,09
15	7,5
18,75	8,59
22,5	9,38
26,25	9,84

30	10

Table 1 - Coordinates of axis parabola

Table 2 lists the elastic weights W of each element, the coordinates of the centers, the static moments S_x of the elastic weights with reference to axis x. The theorem of Varignon allows to determine the elastic center G of the arch: It is the point of coordinates (30.00 , 7.164). This point is assumed as the origin of the new reference system with axes parallel to the old ones , x_0,y_0

Segment	**W[m^{-3}]**	**Y[m]**	**Sx[m^{-2}]**
1	8,6984	1,17	10,1673000
2	9,2626	3,36	31,122336
3	9,877	5,23	51,656971
4	10,655	6,80	72,454000
5	11,572	8,04	101,36028
6	12,607	8,98	113,21086
7	14,045	9,61	134,972450
8	15,718	9,90	155,608200
	92,434		**670,552397**

Table 2 – Centre of the elastic weight

The arch is built in the abutments: then we apply the equations 10), the unknowns are uncoupled each other : .
The arch and the load conditions are symmetric, then the vertical reaction is a determinate one. In Table 3 every information relating to each elements of the arch is collected, the height h of the section, its area A, its moment of inertia J, the length of each elements Ds, its weight Pp, the weight of the beam connections between the arches three in number, the weight of the pillars, the weight of the slab, 0.2 m thick, the weight of the main I deck beams, of the carriageway, the weight of the deck crosspiece beams, the weight of the road paving, the live loads. according to the rules in force in the period of construction.

Segment	**h [m]**	**A [m^2]**	**J [m^4]**	**Δs [m]**	**Weight arc [N]**	**Beam connections [N]**	**Deck crosspiece beams [N]**	**pillars [N]**	**Weight deck [N]**	**Live load [N]**	**Total [kN]**
1	1,968	1,574	0,5081	4,42	173970	100000	62500	82800	178120	187500	784,90
2	1,905	1,524	0,4609	4,269	162650	0	62500	61400	178120	187500	652,17

3	1,843	1,474	0,4173	4,122	151940	100000	62500	43100	178120	187500	723,16
4	1,780	1,424	0,3760	4,006	142610	0	62500	28000	178120	187500	598,74
5	1,717	1,374	0,3375	3,905	134100	100000	62500	15800	178120	187500	678,02
6	1,658	1,327	0,3040	3,832	127080	0	62500	6700	178120	187500	561,91
7	1,592	1,274	0,2690	3,778	120290	0	62500	0	178120	187500	548,42
8	1,530	1,224	0,2388	3,753	114840	0	62500	0	178120	187500	542,97
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1	1,968	1,574	0,5081	4,42	173970	100000	62500	82800	178120	187500	784,90

Table 3 –Arch load analysis

Segment	**W_i[m-^{3}]**
1	8,69838
2	9,26259
3	9,87696
4	10,6547
5	11,5718
6	12,6068
7	14,0451
8	15,7179
8	15,7179
7	14,0451
6	. 12,6068
5	11,5718
4	10,6547
3	9,8770

2	9,2626
1	4,3492

Table 4 Elastic weights

With the following table 5 we determine the first unknown m_G. We apply the first equation of 10).

The first column of table 5, shows the segments from the first left to the last on the right .In the second you suppose the load (from table 3) applied on segment 1 at left and every segment, one after the other elastic, from the first at right to the first at left .Apply the Culmann's third theorem, multiply by the load and sum up all terms of the column. In the third column the applied load is on segment 2, repeat for the segment 3 and the following to the last load at right .
Sum up all results and obtain the final static moment and the final rotation of the terminal section of the beam loaded as in table 3.

Tronchi	Distanza forza (1)	Sp$_i$(1) [m-^{2}]	Distanza forza (2)	Sp$_i$(2) [m-^{2}]	Distanza forza (3)	Sp$_i$(3) [m-^{2}]	Distanza forza (4)	$_i$(4) [m-^{2}]	Distanza forza (5)	Sp$_i$(5) [m-^{2}]
1	56,25	489,28	52,5	456,66	48,75	424,05	45	391,43	41,25	358,81
2	52,5	486,29	48,75	451,55	45	416,82	41,25	382,08	37,5	347,35
3	48,75	481,5	45	444,46	41,25	407,42	37,5	370,39	33,75	333,35
4	45	479,46	41,25	439,51	37,5	399,55	33,75	359,60	30	319,64
5	41,25	477,34	37,5	433,94	33,75	390,55	30	347,15	26,25	303,76
6	37,5	472,76	33,75	425,48	30	378,21	26,25	330,93	22,5	283,65
7	33,75	474,02	30	421,35	26,25	368,68	22,5	316,01	18,75	263,35
8	30	471,54	26,25	412,60	22,5	353,65	18,75	294,71	15	235,77
8	26,25	412,6	22,5	353,65	18,75	294,71	15	235,77	11,25	176,83
7	22,5	316,01	18,75	263,35	15	210,68	11,25	158,01	7,5	105,34
6	18,75	236,38	15	189,10	11,25	141,83	7,5	94,55	3,75	47,28
5	15	173,58	11,25	130,18	7,5	86,79	3,75	43,39	0	0
4	11,25	119,87	7,5	79,91	3,75	39,96	0	0	0	0
3	7,5	74,077	3,75	37,04	0	0	0	0	0	0
2	3,75	34,735	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0
		Tot=5199,4		**Tot=4538,79**		**Tot=3912,89**		**Tot=3324,02**		**Tot=2775,11**

Table 5

Tronchi	Distanza forza (6)	Sp$_i$(6) [m-^{2}]	Distanza forza (7)	Sp$_i$(7) [m-^{2}]	Distanza forza (8)	Sp$_i$(8) [m-^{2}]	Distanza forza (8)	Sp$_i$(8) [m-^{2}]	Distanza forza (7)	Sp$_i$(7) [m-^{2}]

1	37,5	326,19	33,75	293,57	30	260,95	26,25	228,33	22,5	195,71
2	33,75	312,61	30	277,88	26,25	243,14	22,5	208,41	18,75	173,67
3	30	296,31	26,25	259,27	22,5	222,23	18,75	185,19	15	148,15
4	26,25	279,69	22,5	239,73	18,75	199,78	15	159,82	11,25	119,87
5	22,5	260,37	18,75	216,97	15	173,58	11,25	130,18	7,5	86,79
6	18,75	236,38	15	189,10	11,25	141,83	7,5	94,55	3,75	47,28
7	15	210,68	11,25	158,01	7,5	105,34	3,75	52,67	0	0
8	11,25	176,83	7,5	117,88	3,75	58,94	0	0	0	0
8	7,5	117,88	3,75	58,94	0	0	0	0	0	0
7	3,75	52,67	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0
		Tot=2269,60		**Tot=1811,36**		**Tot=1405,79**		**Tot=1059,16**		**Tot=771,47**

Tronchi	**Distanza forza (6)**	**Sp$_i$(6) [m-^{2}]**	**Distanza forza (5)**	**Sp$_i$(5) [m-^{2}]**	**Distanza forza (4)**	**Sp$_i$(4) [m-^{2}]**	**Distanza forza (3)**	**Sp$_i$(3) [m-^{2}]**	**Distanza forza (2)**	**Sp$_i$(2) [m-^{2}]**
1	18,75	163,09	15	130,48	11,25	97,86	7,5	65,24	3,75	32,62
2	15	138,94	11,25	104,20	7,5	69,47	3,75	34,73	0	0
3	11,25	111,12	7,5	74,08	3,75	37,04	0	0	0	0
4	7,5	79,91	3,75	39,96	0	0	0	0	0	0
5	3,75	43,39	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0

5	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0
		Tot=536,45		**Tot=348,71**		**Tot=204,36**		**Tot=99,97**		**Tot=32,62**

Tronchi	**Distanza forza (1)**	**$Sp_i(1)$ [m^{-2}]**
1	0	0
2	0	0
3	0	0
4	0	0
5	0	0
6	0	0
7	0	0
8	0	0
8	0	0
7	0	0
6	0	0
5	0	0
4	0	0
3	0	0
2	0	0
1	0	0
		Tot=0

Table 4 – Elastic weights and static moments

Then the first of 10).

the check was carried out by the method of allowable stresses obtaining:

$$\sigma = \pm 19.94 \frac{kg}{cm^2}$$

we solve now to the third of the equations 10).
. In it now is the arch elastic weight centrifugal moments with respect to the x_0-axis and the direction of the force.

Segment	**$W_i[m^3]$**	**Distance from x_0**
1	8,69838	-6,00
2	9,26259	-3,81
3	9,87696	-1,93
4	10,6547	-0,37
5	11,5718	0,88
6	12,6068	1,82
7	14,0451	2,44
8	15,7179	2,75
8	0,7179	2,75
7	14,0451	2,44
6	12,6068	1,82
5	11,5718	0,88
4	10,6547	-0,37
3	9,8770	-1,93
2	9,2626	-3,81
1	4,3492	-6,00

Table 6 Elastic weights and distance from x_0

Table 7 Centrifugal moments

Tronchi	Distanza forza (1)	Jpx_i (1) [m^4]	Distanza forza (2)	Jpx_i (2) [m^4]	Distanza forza (3)	Jpx_i (3) [m^4]	Distanza forza (4)	Jpx_i (4) [m^4]]	Distanza forza (5)	Jpx_i (5) [m^4]
1	56,25	-2935,70	52,5	-2739,99	48,75	-2544,28	45	-2348,56	41,25	-2152,85
2	52,5	-1852,75	48,75	-1720,41	45	-1588,07	41,25	-1455,73	37,5	-1323,39
3	48,75	-929,30	45	-857,81	41,25	-786,33	37,5	-714,84	33,75	-643,36
4	45	-177,40	41,25	-162,62	37,5	-147,83	33,75	-133,05	30	-118,27
5	41,25	420,06	37,5	381,87	33,75	343,68	30	305,50	26,25	267,31
6	37,5	860,42	33,75	774,37	30	688,33	26,25	602,29	22,5	516,25
7	33,75	1156,61	30	1028,10	26,25	899,59	22,5	771,07	18,75	642,56
8	30	1296,73	26,25	1134,64	22,5	972,55	18,75	810,46	15	648,36
8	26,25	1134,64	22,5	972,55	18,75	810,46	15	648,36	11,25	486,27
7	22,5	771,07	18,75	642,56	15	514,05	11,25	385,54	7,5	257,02
6	18,75	430,21	15	344,17	11,25	258,12	7,5	172,08	3,75	86,04
5	15	152,75	11,25	114,56	7,5	76,37	3,75	38,19	0	0,00
4	11,25	-44,35	7,5	-29,57	3,75	-14,78	0	0,00	0	0,00
3	7,5	-142,97	3,75	-71,48	0	0,00	0	0,00	0	0,00
2	3,75	-132,34	0	0,00	0	0,00	0	0,00	0	0,00
1	0	0,00	0	0,00	0	0,00	0	0,00	0	0,00
		Tot= 7,67		**Tot= - 189,06**		**Tot= - 518,14**		**Tot= - 918,70**		**Tot= - 1334,04**

Tronchi	Distanza forza (6)	Jpx_i (6) [m^4]	Distanza forza (7)	Jpx_i (7) [m^4]	Distanza forza (8)	Jpx_i (8) [m^4]	Distanza forza (8)	Jpx_i (8) [m^4]	Distanza forza (7)	Jpx_i (7) [m^4]
1	37,5	-1957,13	33,75	-1761,42	30	-1565,71	26,25	-1369,99	22,5	-1174,28
2	33,75	-1191,05	30	-1058,71	26,25	-926,37	22,5	-794,04	18,75	-661,70
3	30	-571,88	26,25	-500,39	22,5	-428,91	18,75	-357,42	15	-285,94
4	26,25	-103,48	22,5	-88,70	18,75	-73,92	15	-59,13	11,25	-44,35
5	22,5	229,12	18,75	190,93	15	152,75	11,25	114,56	7,5	76,37
6	18,75	430,21	15	344,17	11,25	258,12	7,5	172,08	3,75	86,04
7	15	514,05	11,25	385,54	7,5	257,02	3,75	128,51	0	0,00
8	11,25	486,27	7,5	324,18	3,75	162,09	0	0,00	0	0,00

8	7,5	324,18	3,75	162,09	0	0,00	0	0,00	0	0,00
7	3,75	128,51	0	0,00	0	0,00	0	0,00	0	0,00
6	0	0,00	0	0,00	0	0,00	0	0,00	0	0,00
5	0	0,00	0	0,00	0	0,00	0	0,00	0	0,00
4	0	0,00	0	0,00	0	0,00	0	0,00	0	0,00
3	0	0,00	0	0,00	0	0,00	0	0,00	0	0,00
2	0	0,00	0	0,00	0	0,00	0	0,00	0	0,00
1	0	0,00	0	0,00	0	0,00	0	0,00	0	0,00
		Tot=-1711,20		**Tot=-2002,32**		**Tot=-2164,92**		**Tot=-2165,43**		**Tot=-2003,85**

Tronchi	**Distanza forza (6)**	**Jpx_i (6) [m^4]**	**Distanza forza (5)**	**Jpx_i (5) [m^4]**	**Distanza forza (4)**	**Jpx_i (4) [m^4]**	**Distanza forza (3)**	**Jpx_i (3) [m^4]**	**Distanza forza (2)**	**Jpx_i (2) [m^4]**
1	18,75	-978,57	15	-782,85	11,25	-587,14	7,5	-391,43	3,75	-195,71
2	15	-529,36	11,25	-397,02	7,5	-264,68	3,75	-132,34	0	0,00
3	11,25	-214,45	7,5	-142,97	3,75	-71,48	0	0,00	0	0,00
4	7,5	-29,57	3,75	-14,78	0	0,00	0	0,00	0	0,00
5	3,75	38,19	0	0,00	0	0,00	0	0,00	0	0,00
6	0	0,00	0	0,00	0	0,00	0	0,00	0	0,00
7	0	0,00	0	0,00	0	0,00	0	0,00	0	0,00
8	0	0,00	0	0,00	0	0,00	0	0,00	0	0,00
8	0	0,00	0	0,00	0	0,00	0	0,00	0	0,00

7	0	0,00	0	0,00	0	0,00	0	0,00	0	0,00
6	0	0,00	0	0,00	0	0,00	0	0,00	0	0,00
5	0	0,00	0	0,00	0	0,00	0	0,00	0	0,00
4	0	0,00	0	0,00	0	0,00	0	0,00	0	0,00
3	0	0,00	0	0,00	0	0,00	0	0,00	0	0,00
2	0	0,00	0	0,00	0	0,00	0	0,00	0	0,00
1	0	0,00	0	0,00	0	0,00	0	0,00	0	0,00
		Tot= -1713,76		**Tot= -1337,62**		**Tot= -923,30**		**Tot= -523,77**		**Tot= -195,71**

Tronchi	**Distanza forza (1)**	**Jpx_i (1) [m^4]**
1	0	0
2	0	0
3	0	0
4	0	0
5	0	0
6	0	0
7	0	0
8	0	0
8	0	0
7	0	0
6	0	0
5	0	0
4	0	0
3	0	0
2	0	0
1	0	0
		Tot=0

Table 7 Centrifugal moments of elastic weights

Applying the third of 10) we obtain:

H=10312775 N

1.3 Carlo Alberto Castigliano [4]

Castigliano states in 1875, in his paper *Nuova Teoria intorno all'equilibrio dei sistemi elastici* presented alla Reale Accademia della Scienza a Torino his famous theore, *the partial derivative of the work of deformation, expressed as a function of external forces, made respect to any of those forces , is equal to the displacement of its point of application*.
He applies the theorem to the study of the constraint reactions of many statically indeterminate structures and in particular to the stone arch on the river Dora, also known as Bridge Mosca, appointed after its illustrious designer.

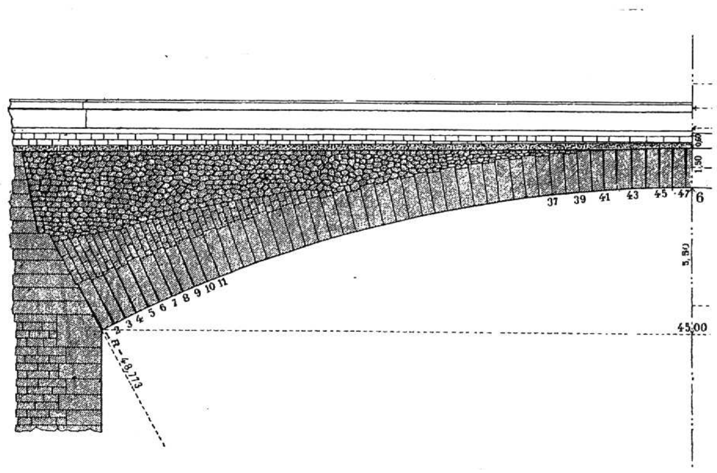


Figure 1 Geometry of Bridge Mosca at Turin

The arch, figure 1,is a voussoir circular segmetal bridg : voussoirs are made with blocks of granite obtained from a quarry near Pinerolo (Torino). The span is 45 m, while its rise is 5.5 m, $\rho = \frac{1}{8.3}$. The radius of the arch is at the intrados 48,773 m and its central angle is 54 ° 56 '42''. The thickness of the rib in the key is 1.5 m, while at abutments is 2 m. The number of stone blocks is 95.
Above the stone work, a mortar layer is laid, 0,15 m, and above a layer of infill. The pavement was in macadam.
Castigliano reduces the weights of the various materials to the weight of a single material, the granite, as shown in figure 2.

[4] Carlo Alberto Castigliano Théorie de l' Equilibre des Systèmes Elastiques et ses Applications", 1879

1.4.

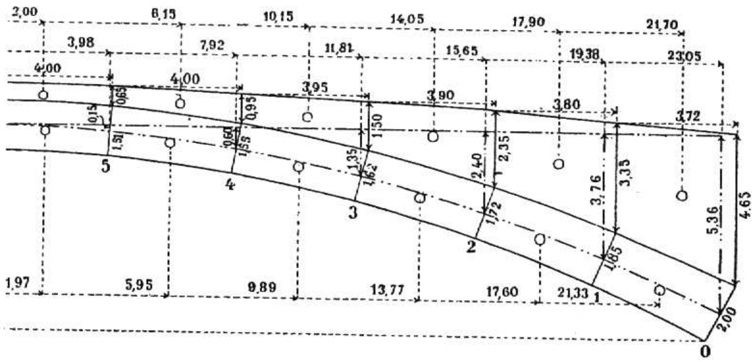


Figure 2. All different materials of the bridge are made comparable by Castigliano

The centroidal axis of the arch is divided into six elements of equal length. Castigliano presumes the arch to be built in the abutments even if the material has no tensile strength. Then he supposes: *if the arch is appropriately designed, the pressure is distributed on the whole section, then there is no parting of points of the section from supports*.
Since the arch is symmetric he assumes the indeterminate unknowns to be the thrust H and the bending moment C in the keystone . Then Castigliano determines bending moment, normal force, as shown in the following table, as a function of the statically indeterminate unknowns, just listed and the weights, in correspondence of the divisions .

Section	Bending moment kgm	Normal force kg	Shear
0	M_0=C-5.36H+2021937	N_0=0.895H+102217	T_0=0.447H-204655
1	M_1=C-3.76H+1318295	N_1=0.925H+ 63085	T_1=0.378H-154385
2	M_2=C-2.40H+ 800470	N_2=0.950H+ 35915	T_2=0.305H-111897
3	M_3=C-1.35H+ 429935	N_3=0.973H+ 18177	T_3=0.230H- 76890
4	M_4=C-0.60H+ 185955	N_4=0.988H+ 7480	T_4=0.154H- 47987
5	M_5=C-0.15H+ 46392	N_5=0.996H+ 1815	T_5=0.078H- 22962
6	M_6=C	N_6=1.000H	T_6=0

Table 1 Bending moment, normal force, shear in the arch. Units Units of measurement are in agreement with the original text

He assumes bending moment C positive if counterclockwise, normal force H positive if compressive.

Cross section	Area m	Moment of inertia m
0	2.01	0.67672
1	1.85	0.52764
2	1.72	0.42404
3	1.62	0.35429
4	1.55	0.31032
5	1.51	0.28691
6	1.50	0.28125

Table 2 Areas and moments of inertia of arc sections

Disregarding shear deformation the Internal Work:

$$L_i = 2\frac{4.00}{2E}\left(\sum\frac{M^2}{I}+\sum\frac{N^2}{A}\right)$$

From the tables 1 and 2 we obtain:

$$L_i = 2\frac{4.00}{2E}\left(16.34C^2 - 22.98\cdot 2CH + 70.06H^2 + 7818893\cdot 2C - 1466630\cdot 2H\right):$$

Castigliano notes that due to the symmetry the section key does not rotate, and its horizontal displacemen is zero.He applies his theorem and equates to zero the derivatives of L_i with respect to C and H, he obtains the two equations:

$$16.34C - 22.98H - 7818893 = 0$$

$$-22.98C + 70.06H - 23249885 = 0$$

and the solution:

$$C = -21800$$

$$H = 324710$$

When you know the statically indeterminate reactions it is easy to determine stresses in sections 0,1,2,3,4,5,6. by table 1 and by formulas of Navier on bending the system of normal stresses in all sections is achieved.

In section 0, abutments, we obtain:

$$M_0 = 259737$$

$$N_0 = 3928272$$

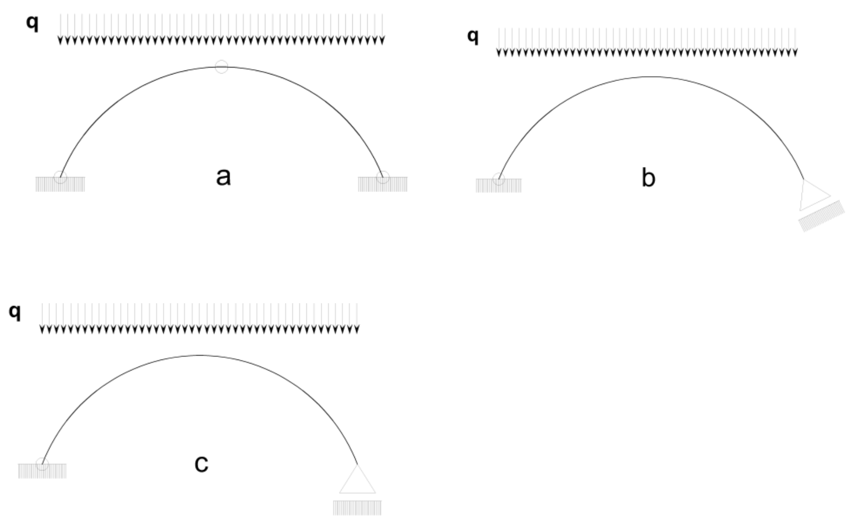
Eccentricity 0.66 m $> \frac{1}{3}$ m, length of half core of inertia of the rectangular section

Castigliano draws then a new centroidal axis, 0.75 m above the arch intradoss changing internal forces , areas , inertia , finds the new eccentricity of the curve of pressures and at abutments he finds 0,30 m .But 1,50/6=0,25m < 0,30m .So the curve of pressare is again outside the core of inertia..
Since the depth of the compression zone is 1.37 m, you can iterate a new approximation assuming the compressed area of 1.37 m. In this case the maximum compression would be approximately equal to 60 $\frac{kg}{cm^2}$, fully acceptable for the granite.

3.4 Mörsch, The arc funicular of loads[5]

Let us consider a parabolic arch, and let us suppose that it is loaded by a constant horizontal unit load. We know that the parabola is the curve funicular of this load. This curve needs three constants ,then if the arc is statically determinate and constraints are right fitted, normal forces only are active in the arch: thìs arc is known as *funicular of loads.*

Compare ,e.g., the three statically determinate structures of figure 1; their centroidal axes are parabolas, but they have different constraints. It is evident that only cases a and b have constraints fitted to normal forces. case c is unfitted, because the reaction of the support is not tangent to the centroidal axis
Then statically determinate arcs funicular of loads depend on its shape, the loading, the boundary conditions



If the arc is statically indeterminate and elastic (double-hinged or fixed ends arcs), you can choose as reduntant reactions H (hinged arch) and the reactive couple m (fixed ends)
Note that if the arc is two hinged, H is applied along hinges, if the arch is built in, direction H is along the principal axis of ellipse .
Let us suppose a two hinged arc, assume e.g. the structure b of figure 1, H is the reduntant reaction . The determinate arc is a funicular of loads. Then each element ds of the arch shortens:

$$\Delta ds = -\frac{Nds}{EA}$$

If θ is the angle of local tangent to arc and the horizontal , and we apply paragraph

1.5. [5] E. Mörsch *Berechnung von Eingespannten Gewölben*. Schweizerische Bauzeitung 1906 seit 83-89.

the integral :

$$\Delta = -\int_l \frac{N\cos\theta ds}{EA} = \frac{H}{E}\int \frac{ds}{A}$$

is the shortening of the span l without any rotation of end sections of the arc

So an *additional thrust* H_A opposite to H must arise within the arc such that :

$$\int \frac{H_A y^2 ds}{EJ} = \frac{H_A}{E}\int \frac{y^2 ds}{J} = \Delta$$

If J is constant the integral represents the moment of inertia of the centroidal axis of the arc relative to the axis x.

If the arc is fixed in the abutments , H_A acts along the axis of ellipse not generating any rotation of end sections of ellipse.

$$\int \frac{H_A {y_0}^2 ds}{EJ} = \frac{H_A}{E}\int \frac{{y_0}^2 ds}{J} = \Delta$$

The relation shows that generally H_A is small in comparison to H:indeed it depends on bending moment in comparison with Δ which depends on normal force .

Then we have shown that the indeterminate arch cannot be a funicular of loads , but the correption H_A is however small .

Now let's show how to design an arch with the procedure of Morsch.
Let's draw the centroidal axis of the arc, assigning a symmetrical parabola that is the funicular of the load q; constant loads are assumed. if it is a bridge they include the weight, the weight of the deck, the live loads; Its geometry, span and rise , takes into account the valley to cross. Assume the arc to be statically determinate, for instance the arc of figure 1, b, divide it into segments following the method of the ellipse of elasticity , trace the funicular of loads. Naturally the funicular deviates from the centroidal axis parabolic. If this shift is acceptable, for example because, according to Culmann, the funicular is always within the core of inertia of each cross section of the beam and therefore the arch is everywhere compressed, the procedure stops; on the contrary case, let's take into account the funicular curve, design a new bridge, verify the new funicular.

If the arch is statically indeterminate and elastic (two-hinged arch or fixed ends) follow the former procedure, choose a determinate arc you can choose as reduntant reactions H (hinged arch) and the reactive couple m (fixed ends) . Apply equation 1) if the arc is two hinged, equation 2) if built in, find H_A with the theory of ellipse of elasticity .

3-19th and 20th century elastic arches.

3.5 .Heyman[6,7,8].Plastic design

In recent years, Heyman has gained fame among the scholars who deal with historical problems associated with the arcs of blocks of stone; indeed there is a breakthrough compared to the past, as he applies the plasticity theorems to this topic.
His most known text on the subject is *The Masonry Arch* of 1982.
He starts from a criticism of the late nineteenth century model of elastic arch, demonstrating that this model is not satisfactory respect of this structure, both as regards the constraint conditions, and for the materials, not resistant to traction, and for the movements that affect its static which often cannot be considered infinitesimal, as required by the theory. He recovers instead the previous theories based on the equilibrium and in particular the theory of Coulomb,.
The mechanical model that Heyman takes into account is rigid-plastic. The assumptions , essentially identical to those of Coulomb, are as follows:
The collapse for relative slipping between two successive ashlars cannot happen.
The masonry has no tensile strength
The stone has a infinite compressive strength
The first hypothesis was already contained in the memory of Coulomb, as he warned that for the materials commonly used in building, friction is so high that this type of failure was ruled out
Even the second goes back to Coulomb and provides that voussoirs are resting one on another .
Coulomb eventually took into consideration the strength of the mortar only to the sliding and not to the tilting over. The hypothesis that the mortar had tensile strength was considered by Navier.
The third hypothesis, is also contained in the theory of Coulomb, the hypothesis is intended to simplify the model and is justified by the fact that in practical cases the compression stress in the arch is much smaller than the rupture of the stone.

The rupture of a masonry arch takes place by tilting around the edges of voussoirs . If N is the normal force trasmitted by a *plastic hinge* m, moment is $M = \pm N h$ (figure 1) *The plastic domain* is shown in the figure 2.

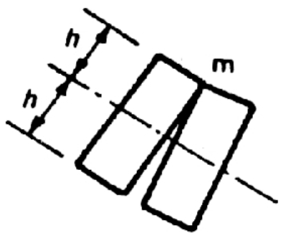


Figure 1 . A plastic hinge between two voussors

It follows that statically admissible is any point within the domain, and then, as it was known up from La Hire, the curve of pressure must be within the masonry or be tangent to extrados or intrados in each point as point m in figure 1. .
The argument continues with the statement of the theorem of plasticity (Lower Bound): *"if you can find a curve of pressure that lies inside the masonry, the arch is safe"*

[6] Heyman, J. "*The Masonry Arch",* Chichester, Ellis Horwood, 1982
[7] Heyman, J. "*Equilibrium of Shell Structures*" (Oxford Engineering Science), Oxford University Press, 1977
[8] Heyman, J. "*The Stone Skeleton*", Int. Journal Solids and Structures, 2, 1966

3-19th and 20th century elastic arches.

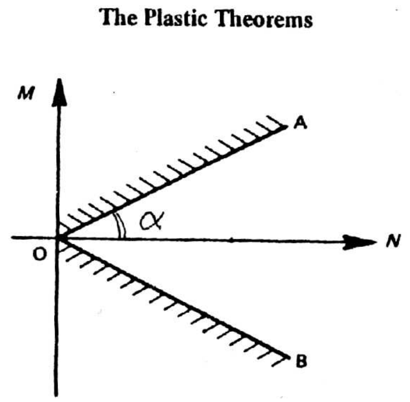


Figure 2. The plastic domain

Unlike the collapse mechanisms applied to ductile structures, which provide a multiplier of loads, and therefore introduce a *factor of safety with respect to the work-loads*, **Heyman suggests a geometric safety factor:** ***the amount by which the actual arch must shrunk to reach its thinnest possible state ,i.e. the amount of which the arc must be reduced to achieve the minimum thickness ,*** **compatibly with arc equilibrium that is, such that the curve of the pressure is still contained within the arc (Figure 3).**

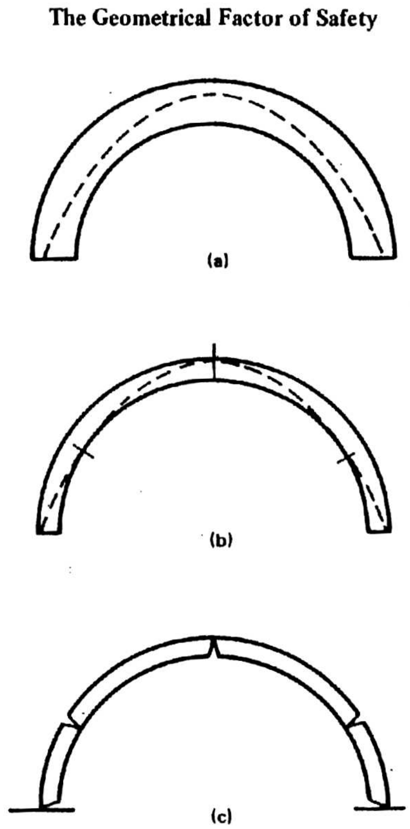


Fig. 3

The three structures of Figure 3 show schematically an application of the idea of geometrical safety factor; from the situation a, it switches to schema b, in which the intrados and the extrados of the arc are reduced to the formation of a collapse mechanism represented in c, with five hinges .
It is proposed a rapid method of solution.
Consider inside the arc real, not represented, an arc reduced, outlined in dashed lines, in a certain ratio. We study this new arc: with reference to the load p .

3-19th and 20th century elastic arches.

Draw the curve of pressure as shown in Figure 4 (compare for example Culmann and Navier), find the unknowns H, p, v_A and v_D. You have four equations in four unknowns :

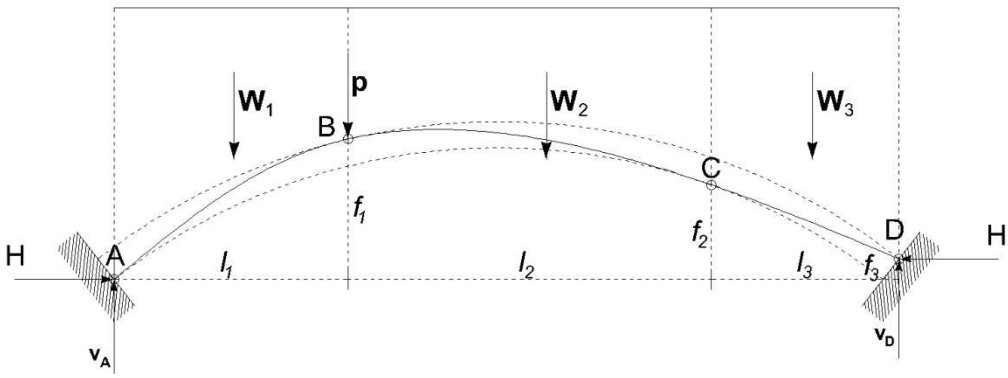


Figura 4

$$v_A + v_D - p - (W_1 + W_2 + W_3) = 0$$

$$v_D l_3 - (f_2 - f_3)H - W_3 \frac{l_3}{2} = 0$$

$$f_1 H - v_A l_1 + W_1 \frac{l_1}{2} = 0$$

$$v_A (l_1 + l_2) + f_2 H + l_2 p + W_3 \frac{l_2}{2} + W_1 \left(\frac{l_1}{2} + l_2 \right) = 0$$

Soving:

$$H = \frac{l_2 \left(\frac{W_2 + W_3}{2} \right)}{f_2 - f_1 + \frac{l_2}{l_3}(f_2 - f_3)}$$

$$v_A = \frac{f_1}{l_1} \left[\frac{l_2 \left(\frac{W_2 + W_3}{2} \right)}{f_2 - f_1 + \frac{l_2}{l_3}(f_2 - f_3)} \right] + \frac{W_1}{2}$$

$$v_D = \frac{f_2 - f_3}{l_3} \left[\frac{l_2 \left(\frac{W_2 + W_3}{2} \right)}{f_2 - f_1 + \frac{l_2}{l_3}(f_2 - f_3)} \right] + \frac{W_3}{2}$$

$$p=\left[\frac{l_2\left(\frac{W_2+W_3}{2}\right)}{f_2-f_1+\frac{l_2}{l_3}\left(f_2-f_3\right)}\right]\left[\frac{f_1}{l_1}+\frac{f_2-f_3}{l_3}\right]-\frac{W_1}{2}-\frac{W_3}{2}-W_2\,.$$

Find the unknowns, draw the polygon of pressures and verify that it is actually inside the arc. If the load is less than or equal to p, the safety coefficient geometric can be assumed greater than or equal to 1. If the load is very different, you must look for a new solution.

Summary