4 SHELLS

4.1 Differential geometry of surfaces

4.1.1 Surfaces. Gaussian surfaces and coordinates in a three-dimension Euclidean space

In a three-dimensional Euclidean space, let a Cartesian coordinate frame be an orthonormal base $(\mathrm{O},\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3)$: It is known as the extrinsic frame of reference.
The locus of points ***P*** *or vectors position,* ***x,*** *such that*, figure 1:

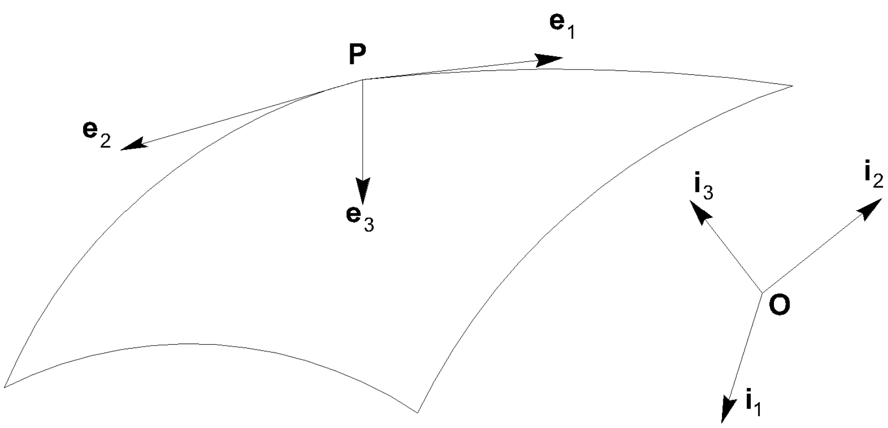

Fig.1 Intrinsic and extrinsic Cartesian frames for a Gaussian surface

$$\mathbf{P}=\mathbf{x}=\mathbf{x}(u^1,u^2)=x^1\mathbf{i}_1+x^2\mathbf{i}_2+x^3\mathbf{i}_3=x^i\mathbf{i}_i \quad \text{, or} \qquad 1)$$

$$\mathbf{x}=\mathbf{x}(u^\alpha,u^\beta) \qquad \alpha,\beta=1,2$$

is a Gaussian surface S: u^1 *,* u^2 *are the curvilinear or Gaussian coordinates*; **x** is a regular , one to one, invertible function so that 1) can be inverted :

$$u^\alpha=u^\alpha(\mathbf{x}) \quad 2)$$

.
If $\overline{u^1}$ and $\overline{u^2}$ are constants $u^1=\overline{u^1}$ and $u^2=\overline{u^2}$ are two families of curves on the surface.
$\mathbf{x}_{,\alpha}=\mathbf{e}*_\alpha$ *is a tangent vector to line* u^α (comma means derivative respect to a variable as usually) and $\mathbf{x}_{,\beta}=\mathbf{e}*_\beta$ *is a tangent vector to line* u^β*.*

The unit vector, **m***, normal to S in the point P, is the vector product:*

$$\mathbf{m}=\frac{\mathbf{e}^*_1\times\mathbf{e}^*_2}{|\mathbf{e}^*_1\times\mathbf{e}^*_2|} \qquad 3)$$

the locus of points **y**

$$\mathbf{y}=\mathbf{x}+h_\alpha\,\mathbf{e}*_\alpha \qquad 4)$$

Is *the tangent plane i*n **x**

The locus of points **y**:

$$\mathbf{y} = \mathbf{x} + \zeta \mathbf{m} \quad 5)$$

$-\infty < \zeta < +\infty$

is the normal line in P to the surface S;
The tangent plane in point P can be represented also with the scalar product :

$$(\mathbf{y} - \mathbf{x}) \cdot \mathbf{m} = 0 \quad 6)$$

a relation which is obtained from 5) through a scalar product if h=0.

Base (P,$\mathbf{e}_1$, $\mathbf{e}_2$) is covariant: the contravariant basis is (P ,$\mathbf{e}^1$, $\mathbf{e}^2$):

7) (P $\mathbf{e}^{*\alpha}$ $\mathbf{e}^{*\beta}$)

such that :

$$\mathbf{e}^{*\alpha} = \frac{\partial u^\alpha}{\partial \mathbf{x}} \qquad \mathbf{e}^{*\beta} = \frac{\partial u^\beta}{\partial \mathbf{x}}$$ and:

$$\mathbf{e}^{*\alpha} \cdot \mathbf{e}^{*\beta}{}_{\beta} = \delta^\alpha_\beta$$

The physical basis is defined as:

$$|\mathbf{e}*_\alpha|^{-1} \mathbf{e}*_\alpha = \mathbf{e}_\alpha$$

In Figure 1 is also represented the intrinsic reference frame at a point P(**P,**$\mathbf{e}_1$,$\mathbf{e}_2$ $\mathbf{e}_3$): they are the unit vectors tangent to the lines $u^1 = \overline{u^1}$ and $u^2 = \overline{u^2}$ $\mathbf{e}_1$ and e_2 and by the unit normal **m** to the surface so as to have a right-handed orthonormal base .

4.1.2 Curves on a surface .

Let $u^1 = u^1(t)$ e $u^2 = u^2(t)$ be two functions of a parameter t, then:

$$\mathbf{x} = \mathbf{x}\left(u^1(t), u^2(t)\right) \quad 1)$$

is a curve C on S (**1.1.1)**
Then:

$$\mathbf{x}' = \frac{d\mathbf{x}}{dt} = \mathbf{x}_{,\alpha} \frac{du^\alpha}{dt} \quad 2)$$

Is a vector tangent to the curve .
If s is the length of curve C the unit tangent vector **t** is:

$$\mathbf{t} = \dot{\mathbf{x}} = \frac{d\mathbf{x}}{dt}\frac{dt}{ds} = \frac{\mathbf{x}'}{|\mathbf{x}'|} \quad 3)$$

4.1.2.1 The first quadratic form. The Gauss tensor A

Let's consider on S the points P and P+dP and the vector d**P**
The square of the vector :

$$d\mathbf{P}\cdot d\mathbf{P}=ds^2=d\mathbf{x}\cdot d\mathbf{x}=\mathbf{x}_{,\alpha}\cdot\mathbf{x}_{,\beta}\,du^\alpha du^\beta=A_{\alpha\beta}du^\alpha du^\beta=\mathbf{A}\,d\mathbf{u}^\alpha\,d\mathbf{u}^\beta=$$
$$=A_{11}\left(du^1\right)^2+A_{22}\left(du^2\right)^2+2A_{12}du^1du^2 \quad (1)$$

$$ds^2=d\mathbf{x}\cdot d\mathbf{x}=\mathbf{e}^*{}_\alpha\cdot\mathbf{e}^*{}_{,\beta}\,du^\alpha du^\beta=A_{\alpha\beta}du^\alpha du^\beta=\mathbf{A}\,d\mathbf{u}^\alpha\,d\mathbf{u}^\beta=$$

$A_{\alpha\beta}du^\alpha du^\beta$ *is the first surface Gaussian quadratic form*.

$d\mathbf{u}^\alpha\,d\mathbf{u}^\beta$ is a tensor vector product, ds^2 is a scalar, then $A_{\alpha\beta}$ is a second order covariant symmetric tensor, known as *Gauss tensor*, defined in the two dimensions surface S as:

$$\mathbf{x}_{,\alpha}\cdot\mathbf{x}_{,\beta}=\mathbf{e}^*_\alpha\cdot\mathbf{e}^*{}_\beta=\mathbf{A} \quad (2)$$

Tensor $A_{\alpha\beta}$ is definite positive, indeed $ds^2>0$. It has in the two dimensions space S, three independent components named after Gauss as E,F,G:

$$A_{11}=E=\mathbf{x}_{,1}\cdot\mathbf{x}_{,1}=\mathbf{e}^*{}_1\cdot\mathbf{e}^*{}_1=\left|\mathbf{e}^*{}_1\right|^2>0 \quad 3)$$

$$A_{22}=G=\mathbf{x}_{,2}\cdot\mathbf{x}_{,2}=\mathbf{e}^*{}_2\cdot\mathbf{e}_2{}^*=\left|\mathbf{e}^*{}_2\right|^2>0$$

$$A_{12}=A_{21}=F=\mathbf{x}_{,1}\cdot\mathbf{x}_{,2}=\mathbf{e}^*_1\cdot\mathbf{e}^*_2 \quad)$$

Then the length of arc element u^1 is :

$$\sqrt{E}du^1 \quad (4)$$

The length of arc element u^2 :

$$\sqrt{G}du^2$$

Notice:

$$1=A_{\alpha\beta}\frac{du^\alpha}{ds}\frac{du^\beta}{ds}=\mathbf{A}\frac{d\mathbf{u}^\alpha}{ds}\frac{d\mathbf{u}^\beta}{ds}=\mathbf{A}\mathbf{e}_\alpha\otimes\mathbf{e}_\beta \quad 5)$$

$\mathbf{e}^\alpha$ **,** $\mathbf{e}^\beta$ are unit vectors tangent to lines u^1,u^2 .5

The angle α between lines u^1 e u^2 is

$$\cos\alpha=\frac{\mathbf{e}_1}{}\cdot\frac{\mathbf{e}_2}{}=\frac{F}{\sqrt{EG}} \quad 6)$$

Then lines are normal each other if and only if F=0.
Notice::

$$u^\alpha{}_{,\mathbf{x}}\cdot u^\beta{}_{,\mathbf{x}}=A^{\alpha\beta} \quad (7)$$

$$A_{ik}A^{kj}=\delta_i^j \quad (8)$$

And matrix $A^{\alpha\beta}$ is the inverse of $A_{\alpha\beta}$.

$$A^{11} = \frac{A_{22}}{\left|A_{\alpha\beta}\right|},\; A^{12} = A^{21} = -\frac{A_{12}}{\left|A_{\alpha\beta}\right|},\; A^{22} = \frac{A_{11}}{\left|A_{\alpha\beta}\right|} \qquad \textbf{9)}$$

4.1.2.2 - The second quadratic form.The Weingarten tensor B

With reference to point P and to point P+dP of the surface S of figure 1, let's consider vectors **P, P+dP, m, m+dm** .

The scalar product:

$$d(\mathbf{m}\cdot\mathbf{m}) = 2(\mathbf{m}\cdot d\mathbf{m}) = 0$$

Then vectors d**m** and d**x** are both in the tangent plane in P to S. Then:

$$d\mathbf{m} = \mathbf{m}_{,\alpha}\, du^{\alpha} \qquad d\mathbf{x} = \mathbf{x}_{,\beta}\, du^{\beta}$$

Their scalar product :

$$-d\mathbf{m}\cdot d\mathbf{x} = -\mathbf{m}_{,\alpha}\cdot\mathbf{x}_{,\beta}\, du^{\alpha}du^{\beta} = -\frac{1}{2}\left(\mathbf{m}_{,\alpha}\cdot\mathbf{x}_{,\beta} + \mathbf{m}_{,\beta}\cdot\mathbf{x}_{,\alpha}\right)du^{\alpha}du^{\beta} = B_{\alpha\beta}du^{\alpha}du^{\beta} = \mathbf{B}\, d\mathbf{u}^{\alpha}\, d\mathbf{u}^{\beta} \qquad (1\,)$$

Then $B_{\alpha\beta}$ is a second order covariant symmetric tensor known as Weingarten tensor, defined in the two dimensions shell space. Its components are the following:

$$B_{11} = L = -\mathbf{x}_{,1}\cdot\mathbf{m}_{,1} = \mathbf{x}_{,11}\cdot\mathbf{m} \qquad 2)$$

$$B_{22} = N = -\mathbf{x}_{,2}\cdot\mathbf{m}_{,2} = \mathbf{x}_{,22}\cdot\mathbf{m}$$

$$B_{12} = M = -\frac{1}{2}\left(\mathbf{x}_{,1}\cdot\mathbf{m}_{,2} + \mathbf{x}_{,2}\cdot\mathbf{m}_{,1}\right) = \mathbf{x}_{,12}\cdot\mathbf{m}$$

E.g. from:

$$\mathbf{x}_{,1}\cdot\mathbf{m} = 0$$

and:

$$\mathbf{x}_{,11}\cdot\mathbf{m} + \mathbf{x}_{,1}\cdot\mathbf{m}_{,1} = 0$$

4.1.2.3 Weingarten and Gauss formulas

Weingarten formulas

$$\mathbf{m}_{,\alpha} = -A^{\beta\gamma}B_{\beta\alpha}\,\mathbf{x}_{,\gamma} \; 1)$$

These formulas are the derivatives respect to u^{α} of the normal vector **m** as function of derivatives of vector $\mathbf{x}_{\alpha}$ and of E,F,G, L,M,N.

E.g.

$$m^{i}_{,1} = \frac{1}{\left|A_{\alpha\beta}\right|}\left(-A^{11}B_{11}x^{i}_{1} - A^{12}B_{11}x^{i}_{2} - A^{21}B_{21}x^{i}_{1} - A^{22}B_{21}x^{i}_{2}\right) = \frac{A^{12}B_{21} - A_{22}B_{11}}{\left|A_{\alpha\beta}\right|}x^{i}_{1} = \frac{FM - GL}{EF - G^{2}}$$

Gauss formula

$$\mathbf{x}_{,\alpha\beta} = \Gamma^k_{\alpha\beta}\,\mathbf{x}_{,k} + B_{\alpha\beta}\,\mathbf{m} \quad 2)$$

2) are the derivatives of vectors $\mathbf{e}_\alpha{}^* = \mathbf{x}_{,\alpha}$ respect to u^α in function of vectors $\mathbf{e}_\alpha$ and of tensors $A^{\alpha\beta}$ e $A_{\alpha\beta}$. , $\Gamma^k_{\alpha\beta}$ is the Christoffel symbol of the second kind . The two symbols, of the first and second kind, are related by:

$$\Gamma^k_{\alpha\beta} = A^{kr}\Gamma_{\alpha\beta r} \quad 3)$$

while:

$$\Gamma_{\alpha\beta k} = \frac{1}{2}\left(\frac{\partial A_{\alpha k}}{\partial u^\beta} + \frac{\partial A_{\beta k}}{\partial u^\alpha} - \frac{\partial A_{\alpha\beta}}{\partial u^k}\right) \quad 4)$$

4.1.2.4 Existence and uniqueness

Let it be six arbitrary functions E,F,G,L,M,N of u^1,u^2, an unique function exists, $\mathbf{x}\left(u^1,u^2\right)$, less than a rigid movement, if the following compatibility equations are satisfied:
Codazzi equations

$$B_{\alpha\beta,\gamma} - B_{\alpha\gamma,\beta} = 0 \quad 1)$$

Two of them are independent

Gauss equation:

$$B_{\rho\beta}B_{\alpha\gamma} - B_{\rho\gamma}B_{\alpha\beta} = R_{\rho\alpha\beta\gamma} \quad 2)$$

R is the Riemann tensor. Only one is independent.

4.1.2.5 Normal curvature

Let P be a point on surface S, $\mathbf{x} = \mathbf{x}\left(u^1,u^2\right)$, C a curve on S through P: $\mathbf{x} = \mathbf{x}\left(u^1(t),u^2(t)\right)$, **t**., its tangent in P.
The vector curvature of C in P , **1.1** 4), of C:

$$\boldsymbol{\kappa} = \frac{d\mathbf{t}}{ds} = \kappa\mathbf{n}$$

Let's define the *vector normal curvature:*

$$\boldsymbol{\kappa}_\mathbf{m} = \left(\boldsymbol{\kappa}\cdot\mathbf{m}\right)\mathbf{m} \quad 1)$$

It is *the vector projection of the vector* $\boldsymbol{\kappa}$ *on the vector* ***m****.*

Notice: $\mathbf{t}\cdot\mathbf{m} = 0$, then:

$$\frac{d\mathbf{t}}{dt}\cdot\mathbf{m} + \mathbf{t}\cdot\frac{d\mathbf{m}}{dt} = 0$$

Then the scalar normal curvature:

$$\kappa_m = \boldsymbol{\kappa}\cdot\mathbf{m} = \frac{d\mathbf{t}}{dt}\cdot\frac{\mathbf{m}}{\left|\frac{d\mathbf{x}}{dt}\right|} = \frac{-\mathbf{t}\cdot\frac{d\mathbf{m}}{dt}}{\left|\frac{d\mathbf{x}}{dt}\right|} = \frac{-\frac{d\mathbf{x}}{dt}\cdot\frac{d\mathbf{m}}{dt}}{\left|\frac{d\mathbf{x}}{dt}\right|^2} = \frac{B_{\alpha\beta}\dot{u}^{\alpha}\dot{u}^{\beta}}{A_{\alpha\beta}\dot{u}^{\alpha}\dot{u}^{\beta}} \qquad 2)$$

As vector **t** varies in the tangent plane in P , the Weingarten tensor takes on all the values between the lower and the higher values of Weingarten tensor and more precisely:

$2K_M$ = traB 3)
K_G =detB

All the curves with tangent **t** in point P have the same normal curvature: indeed the normal curvature depends only from **t** and **m** 1). Then:

$$\kappa_{\mathbf{m}} = \left(\frac{d\mathbf{t}}{ds}\cdot\mathbf{m}\right) = \kappa\mathbf{n}\cdot\mathbf{m} = \kappa\cos\alpha \qquad 4)$$

For *a normal section* (a section through **m**) $\cos\alpha = 1$ and $\kappa_{\mathbf{m}} = \kappa$. 5)

Then the research is limited to normal sections for which $\kappa_{\mathbf{m}} = \kappa$.

4.1.2.6 Maximum and minimum curvature of the normal sections in a point P of a surface

If P is a point of the surface, let it be **m** the normal and the unit tangent vector to P on the surface

$$\lambda^{\alpha} = \frac{du^{\alpha}}{ds} \qquad 6)$$

then:

$$\kappa_m = \frac{B_{\alpha\beta}\lambda^{\alpha}\lambda^{\beta}}{A_{\alpha\beta}\lambda^{\alpha}\lambda^{\beta}} \qquad 7)$$

or the functional Z of vector **t** in P :

$$Z = B_{\alpha\beta}\lambda^{\alpha}\lambda^{\beta} - \kappa_m\left(A_{\alpha\beta}\lambda^{\alpha}\lambda^{\beta}\right) = 0$$

Searching the extremal values of the curvature equal zero its derivative respect to the variable λ^{α} :

$$\left(B_{\alpha\beta} - \kappa_m A_{\alpha\beta}\right)\lambda^{\beta} = 0 \qquad 8)$$

is a linear homogeneous system in λ^{β} which has solution only if the determinant:

$$\left|B_{\alpha\beta} - \kappa_m A_{\alpha\beta}\right| = 0 \qquad 9)$$

obtaining the second degree equation in k_m :

$$\kappa_m{}^2 - 2W\kappa_m + K = 0 \qquad 10)$$

And the autovalues:

$$\kappa_I + \kappa_{II} = 2K_{m\times} \qquad 11)$$

$$\kappa_I \cdot \kappa_{II} = K_G$$

K_m represents the medium curvature ,K_G the Gaussian curvature. To each autovalue corresponds the autovector: they are obtained from equations

$$\left(B_{\alpha\beta} - \kappa_I A_{\alpha\beta}\right)\lambda^I = 0 \qquad 12)$$

$$\left(B_{\alpha\beta} - \kappa_{II} A_{\alpha\beta}\right)\lambda^{II} = 0$$

Se $\kappa_I \neq \kappa_{II}$ both autovectors are orthogonal, as you obtain multiplying the first by λ^α and the second by λ^β and subtracting :

$$\lambda^I \lambda^{II} = 0$$

From 12) :

$$\kappa_I = \frac{B_{11}}{A_{11}} = \frac{1}{R_I} \quad \kappa_{II} = \frac{B_{22}}{A_{22}} = \frac{1}{R_{II}} \qquad 13)$$

R_I e R_{II} are the principal radii of curvature

Se $\kappa_I = \kappa_{II}$, all directions are autovectors .

The sign of eigenvalues determines the classification of the points of a surface in *elliptic points,* characterized by the equality of the sign*, parabolic points*, characterized by an *eigenvalue null* and *hyperbolic points*, characterized by their difference in sign.
A curve on a surface whose tangent in each point is directed along an autovector is called line of curvature. Its ruling equation is:

$$\left(B_{\alpha I} - \kappa_{(I)} A_{\alpha I}\right)\frac{du^I}{ds} = 0 \qquad 14)$$

and

$$\left(B_{\alpha II} - \kappa_{(II)} A_{\alpha II}\right)\frac{du^{II}}{ds} = 0$$

We will review Ruled surfaces, cylindrical, conical, and Revolution surfaces, as spherical: They are the important surfaces in the story of domes

4.1.3 Ruled surfaces

It is a surface generated by a family of straight lines subject to a rule; in the orthonormal base $\left(O, \mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3\right)$ of figure 1 the locus of points **z**:

$$\boldsymbol{z} = \boldsymbol{z}(u^1)$$

is a curve.

$$\mathbf{g} = \mathbf{g}(u^1)$$

is a unit vector ,
The locus of points x:

$$\mathbf{x} = \boldsymbol{z}(u^1) + u^2\,\mathbf{g}(u^1)$$

is a *ruled surface*.
If **g** is a constant vector, the surface is a cylinder :

$$\mathbf{x} = \mathbf{z}(u^1) + u^2\,\mathbf{g}.$$

Le curves $\overline{u^1}$ are translations of C in direction **g**. Le curves $\overline{u^2}$ are straight lines parallel to **g.**

Generally **z** is a curve of plane x_1, x_2.

4.2 The model surface. Definition of shell

Given a surface, named *model* S:

$$\mathbf{x} = \mathbf{x}\left(u^1, u^2\right)$$

let's suppose F=0 and lines $u^1 = \overline{u^1}$ and $u^2 = \overline{u^2}$ to be lines of principal curvature; unit tangents to curvature lines are the vectors $\mathbf{e_1}$ ed $\mathbf{e_2}$, with the vector $\mathbf{m}=\mathbf{e_3}$ they form the intrisic trihedron, the orthonormal intrinsic reference basis. The shell is a continuum: each point **y** of the shell is defined through the equation of the normal line to surface S in a point **x**:

$$\mathbf{y} = \mathbf{x} + \zeta\,\mathbf{m} \qquad 1)$$

ζ is a real variable :

$$-\frac{h}{2} \le \zeta \le \frac{h}{2}$$

h is thickness of the shell . We suppose $h <<$ continuum seizes, then, a shell is mainly a two dimension continuum as a beam is a one dimension continuum. The derivative:

$$\mathbf{y}_{,\alpha} = \mathbf{x}_{,\alpha} + \zeta\,\mathbf{m}_{,\alpha} \qquad 2)$$

Note that with equation 1) for every value $\zeta = \zeta$ a new gaussian surface **y** is obtained, with new Gaussian coordinates v^1, v^2 functions of the old and of ζ.
For one of these surfaces, **y**, let's derive the function respect to v^1, v^2. The scalar product:

$$\mathbf{y}_{,1}\cdot\mathbf{y}_{,2} = \left(\mathbf{x}_{,1} + \zeta\mathbf{m}_{,1}\right)\cdot\left(\mathbf{x}_{,2} + \zeta\mathbf{m}_{,2}\right) = \mathbf{x}_{,1}\cdot\mathbf{x}_{,2} + \zeta\left(\mathbf{m}_{,1}\cdot\mathbf{x}_{,2} + \mathbf{m}_{,2}\cdot\mathbf{x}_{,1}\right) + \zeta^2\,\mathbf{m}_{,1}\cdot\mathbf{m}_{,2}$$

**

shows that the new coordinates are orthogonal, then, $\mathbf{y}_{,1}\cdot\mathbf{y}_{,2} = 0$ everywhere in the shell, not only if the product $\mathbf{x}_{,1}\cdot\mathbf{x}_{,2}$ = L is zero but also if M=0. Indeed if M= $\mathbf{m}_{,1}\cdot\mathbf{x}_{,2} = \mathbf{m}_{,2}\cdot\mathbf{x}_{,1} = 0$, $\mathbf{m}_{,1}$ e $\mathbf{m}_{,2}$ are orthogonal , because both are orthogonal to orthogonal lines and their scalar product is zero. We suppose u^α and u^β to be curvature lines in the model surface , then L=M=0 in the new coordinates 2) and v^α e v^β are also curvature lines.

We will review Ruled surfaces, cylindrical, conical, and Revolution surfaces, as spherical.

Ruled surface
Ruled surfaces are surfaces generated by a family of straight lines subject to a rule.
In the orthonormal base $(O,\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3)$ of figure 1 the locus of points **z**:

$$\boldsymbol{z} = \boldsymbol{z}(u^1)$$

is a curve, the rule, and:

$$\mathbf{g} = \mathbf{g}(u^1)$$

is a unit vector ,
Then the locus of points **x**:

$$\mathbf{x} = \boldsymbol{z}(u^1) + u^2\,\mathbf{g}(u^1)$$

according to the definition, is a *ruled surface*.
If unit vector **g** is constant, the surface is *a cylinder*:

$$\mathbf{x} = \mathbf{z}(u^1) + u^2\,\mathbf{g}.$$

Le curves $\overline{u^1}$ are translations of C in direction **g**. Le curves $\overline{u^2}$ are straight lines parallel to **g.**

Revolution surfaces 0

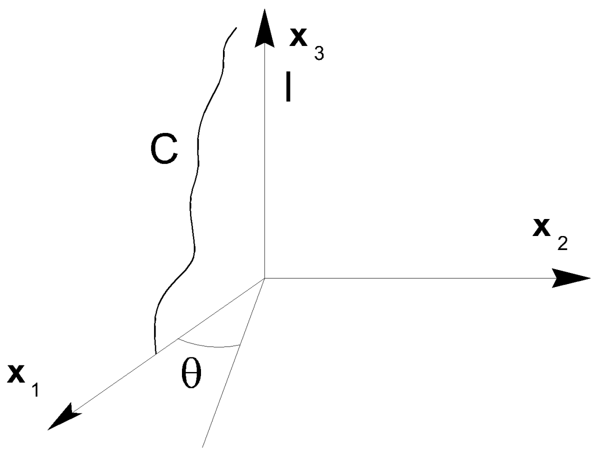

Figure 1

Reolution surfaces are obtained by the rotation of a plane curve C around a fixed line l, in the figure 1 l coincides with the Cartesian axis x_3. The successive positions of the curve C are the meridian curves, the successive positions of each point P of the curve are the parallel circles.
In the Cartesian reference of figure 1 functions :

$$x_1 = x_1(t) \text{ e } x_3 = x_3(t)$$

represent a curve in plane x_1,x_3 .Then:

$$\mathbf{x} = x_1(t)\cos\theta\,\mathbf{i}_1 + x_1(t)\sin\theta\,\mathbf{i}_2 + x_3(t)\,\mathbf{i}_3$$

ìs a point **x** of the surface. Note:

$$F = \mathbf{x}_{,t} \cdot \mathbf{x}_{,\theta} = 0$$

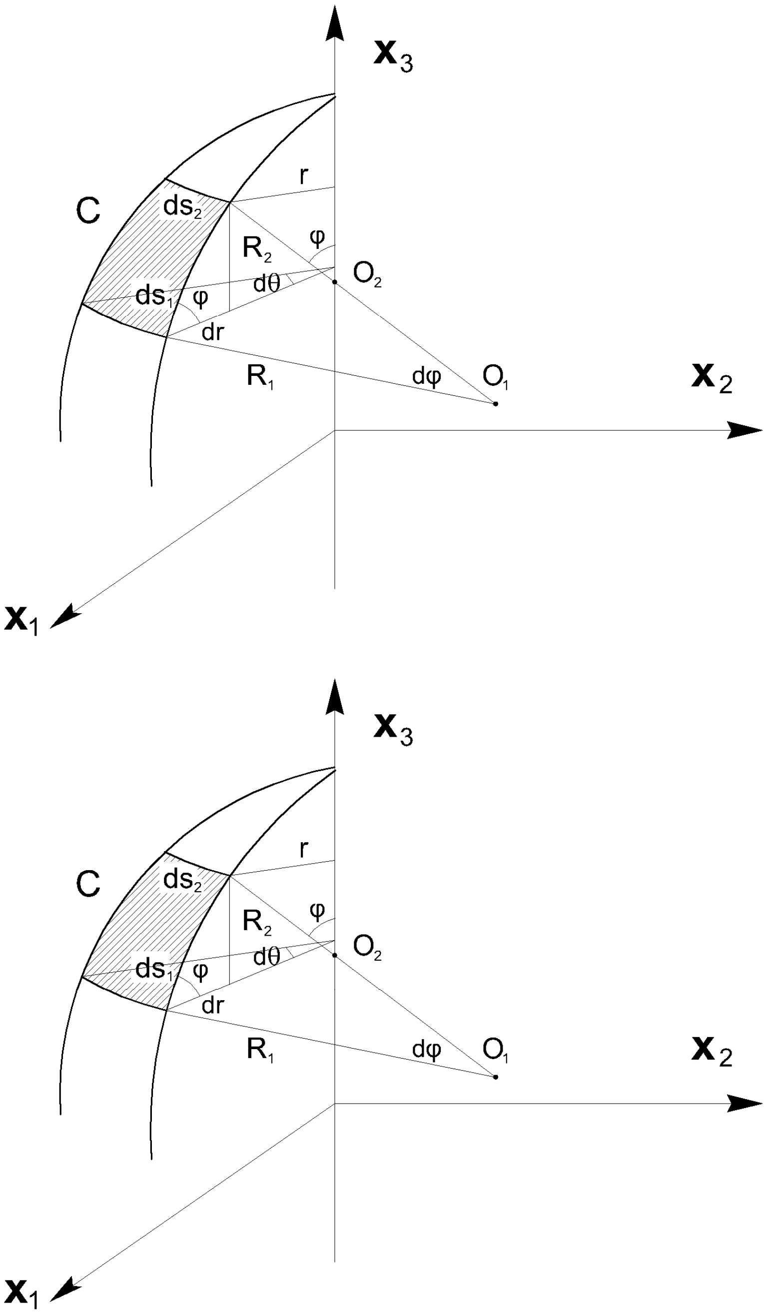

Figure 1 Geometry of a revolution surface

and therefore the curves intersect at right angles. Curves $\overline{\theta}$ are the meridians, curves $\overline{t}$ are the parallels. Each normal **m** in a point P meets l, the meridian plane is a principal plane and the meridian curve is then a line of curvature. The other principal plane is orthogonal, meets **m**, has the same tangent of the parallel; the parallel is the second line of curvature (figure2).

Using spherical coordinates, pose $t = \varphi$, the zenithal angle. If r is the radius of the parallel from, figure 2, we obtain:

$$r = R_2 \sin\varphi$$

$$ds_1 = R_1 d\varphi$$
$$ds_2 = r d\theta = R_2 \sin\varphi d\theta$$
$$dr = ds_1 \cos\varphi = R_1 \cos\varphi d\varphi$$

Then:

$$\sqrt{A_{11}} = R_1$$
$$\sqrt{A_{22}} = R_2 \sin\varphi = r$$

<u>Spherical surface</u>

The equation of the spherical surface in spherical coordinates is:

$$\mathbf{x} = r\sin\varphi\cos\theta\,\mathbf{i}_1 + r\sin\varphi\sin\theta\,\mathbf{i}_2 + r\cos\varphi\,\mathbf{i}_3$$

Then :

$$\mathbf{m} = -\cos\theta\sin\varphi\,\mathbf{i}_1 - \sin\theta\sin\varphi\,\mathbf{i}_2 - \cos\varphi\,\mathbf{i}_3$$

$$A_{11} = \mathbf{x}_{,\theta} \cdot \mathbf{x}_{,\theta} = r^2 \sin^2\varphi$$
$$A_{12} = \mathbf{x}_{,\theta} \cdot \mathbf{x}_{,\varphi} = 0$$
$$A_{22} = \mathbf{x}_{,\varphi} \cdot \mathbf{x}_{,\varphi} = r^2$$

$$B_{11} = \mathbf{x}_{,\theta\theta} \cdot \mathbf{n} = r\sin^2\varphi$$
$$B_{12} = \mathbf{x}_{,\theta\varphi} \cdot \mathbf{n} = 0$$
$$B_{22} = \mathbf{x}_{,\varphi\varphi} \cdot \mathbf{n} = r$$

Then:

$$k_I = k_{II} = \frac{1}{r}$$

It follows that in the spherical surface all directions are eigenvectors .

<u>Conic surface of circular section</u>

Cone is a special case of ruled surface: the generating lines pass through a point C, then with reference to equation:

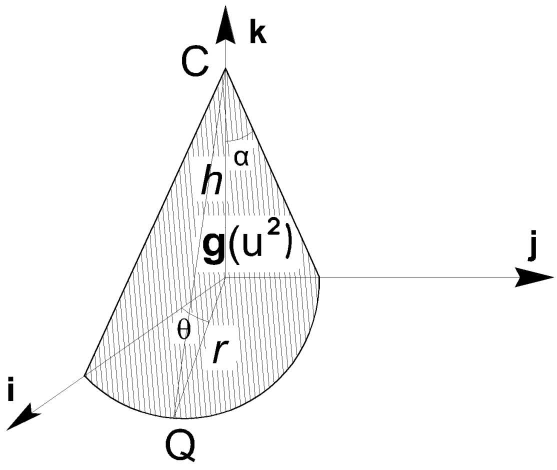

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Figure 1 Conic surface of circular section

$$\mathbf{x} = \boldsymbol{z}(u^1) + u^2\,\mathbf{g}(u^1)$$

Put:

$$v = u^2$$

v is the length of vector P-C, (P is a point on vector Q-C), $0 < v < \left(r^2 + h^2\right)^{\frac{1}{2}} = l$ and:

$$\theta = u^1$$

$$r = h \cdot tg\alpha$$

Vector **g:**

$$\mathbf{g}\left(u^1\right) = \frac{Q - C}{\left|Q - C\right|} = \frac{\left(r \cdot \cos\theta\,\mathbf{i} + r \cdot \sin\theta\,\mathbf{j}\right) - h\mathbf{k}}{l}$$

equation

$$\mathbf{x} = \mathbf{c} + u^2\,\mathbf{g}\left(u^1\right)$$

Then:

$$\mathbf{x} = h\mathbf{k} + \frac{v\left(r \cdot \cos\theta\,\mathbf{i} + r \cdot \sin\theta\,\mathbf{j} - h\mathbf{k}\right)}{l}$$

The derivatives:

$$\mathbf{x}_{,v} = \frac{r \cdot \cos\theta\,\mathbf{i} + r \cdot \sin\theta\,\mathbf{j} - h\mathbf{k}}{l}$$

$$\mathbf{x}_{,\theta} = \frac{v\left(-r \cdot \sin\theta\,\mathbf{i} + r \cdot \cos\theta\,\mathbf{j}\right)}{l}$$

$$G = \mathbf{x}_{,v} \cdot \mathbf{x}_{,v} = 1$$

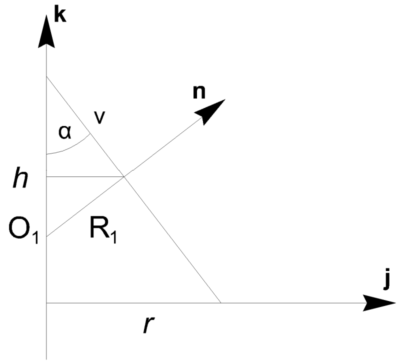
$$E = \mathbf{x}_{,\theta} \cdot \mathbf{x}_{,\theta} = \frac{r^2 v^2}{l^2}$$

$$F = \mathbf{x}_{,v} \cdot \mathbf{x}_{,\theta} = 0$$

Then lines:

$$v = cost$$
$$\theta = cost$$

are orthogonal.
Lines $v = cost$ are parallel circles, lines $\theta = cost$ are the generating lines of the cone.
The two systems are orthogonal: the curvature of the lines θ is zero, the curvature of the lines v is obtained from the figure. Indeed :



$$R_1 = v \cdot tg\alpha$$

The unit vector **m** normal to the conic surface :

$$\mathbf{m} = \frac{\mathbf{x}_{,v} \times \mathbf{x}_{,\theta}}{\left|\mathbf{x}_{,v} \times \mathbf{x}_{,\theta}\right|} = \frac{-r^2 v\,\mathbf{k} + hrv\left(-\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}\right)}{l^3}$$

$$\underline{\mathbf{x}_{,vt}} = 0$$

$$\underline{\mathbf{x}_{,\theta t}}.\mathbf{m} = \frac{+hrv}{l}$$

$$\underline{\mathbf{x}_{,\theta t}}.\mathbf{m} = \frac{+hrv}{l}$$

Equilibrium equations of shells

In Figure 1 is represented the intrinsic reference at a point P(P,**e₁ ,e₂ e₃**) of the shell: these vectors are both the unit vectors tangent to the orthogonal lines of curvature of the surface **e₁ ,e₂** and the unit normal **m=e₃** to the surface, they are a right-handed orthonormal base.

Kirchhoff-Love hypotheses

Theory of shells is based on Kirchhoff-Love hypotheses:

a-The material is linear elastic, homogeneous and isotropic.

b-The fiber normal to the surface, first not deformed, remains normal to the deformed surface and is not subject to dilatation. Therefore:

$$\varepsilon_{33} = \varepsilon_{13} = \varepsilon_{23} = 0$$

c-Stresses σ_{33} are small compared to the others and can be neglected: then for the hypothesis b.

$$\sigma_{33} = \sigma_{13} = \sigma_{23} = 0$$

d-The movements of points of the shell are small compared to its thickness (Love).

Internal forces in shell

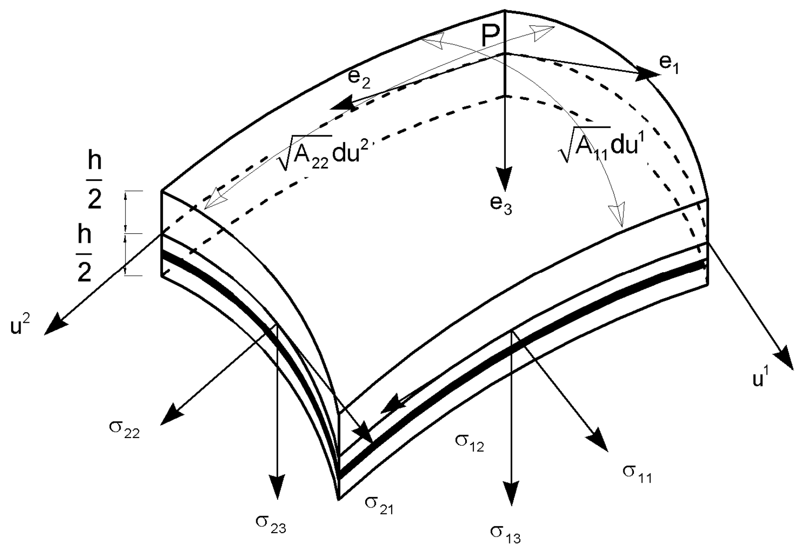


Figure1. Internal forces in shells

Internal forces are defined along the thickness h of the shell. They are defined along its thickness as in the theory of beams.

They are ten. N_{11}, N_{12}, T_{13}, N_{22}, N_{21}, T_{23}, M_{11}, M_{12}, M_{22}, $M_{21,}$ differently from beams they are referred to unit length.

E.g. The internal force N_{11} arises from stress σ_{11}. they are applied on face orthogonal to line u^1 in the principal direction u^1. Then we obtain the following relation. the length of the face parallel to u^2 is, $\sqrt{A_{22}}du^2$, then:

$$N_{11}\sqrt{A_{22}}du^2 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{11}\sqrt{A_{22}}\left(1-\frac{z}{R_2}\right)du^2 dz$$

And likewise

$$N_{12} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{12}\left(1-\frac{z}{R_2}\right)dz$$

$$T_{13} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{13}\left(1-\frac{z}{R_2}\right)dz$$

$$N_{22} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{22}\left(1-\frac{z}{R_1}\right)dz$$

$$N_{21} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{21}\left(1-\frac{z}{R_1}\right)dz$$

$$T_{23} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{23}\left(1-\frac{z}{R_1}\right)dz$$

$$M_{11} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{11} z\left(1-\frac{z}{R_2}\right)dz$$

$$M_{12} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{12} z\left(1-\frac{z}{R_2}\right)dz$$

$$M_{22} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{22} z\left(1-\frac{z}{R_1}\right)dz$$

The internal forces are then ten and more exactly forces are six N_{11}, N_{12}, N_{21}, N_{22}, T_{13}, T_{23} and moments are four $M_{11,}$ M_{22}, M_{12}, M_{21}. Shears T_{13}, T_{23} are determined only by equilibrium equations for the Kirchhoff hypothesis , but not obtained by integration. Note that the symmetry of the stress tensor does not imply the corresponding symmetry $N_{12} = N_{21}$ and $M_{12} = M_{21}$.
The corresponding stresses are obtained by the following formulas with reference to a unit width:

$$\sigma_{11} = \left(\frac{N_{11}}{h} + \frac{12M_{11}}{h^3} z\right)\frac{1}{1-\frac{z}{R_2}} \approx \frac{N_{11}}{h} + \frac{12M_{11}}{h^3} z$$

$$\sigma_{22} = \left(\frac{N_{2}}{h} + \frac{12M_{22}}{h^3} z\right)\frac{1}{1-\frac{z}{R_1}} \approx \frac{N_{22}}{h} + \frac{12M_{22}}{h^3} z$$

$$\sigma_{12} = \frac{1}{2h}\left(N_{12}+N_{21}\right) + \frac{6}{h^3}\left(M_{12}+M_{21}\right)z$$

$$\sigma_{i3} = \frac{T_{i3}}{h}\left(1-\left(\frac{2z}{h}\right)^2\right),\ \ i=1,2$$

Equilibrium of a shell element

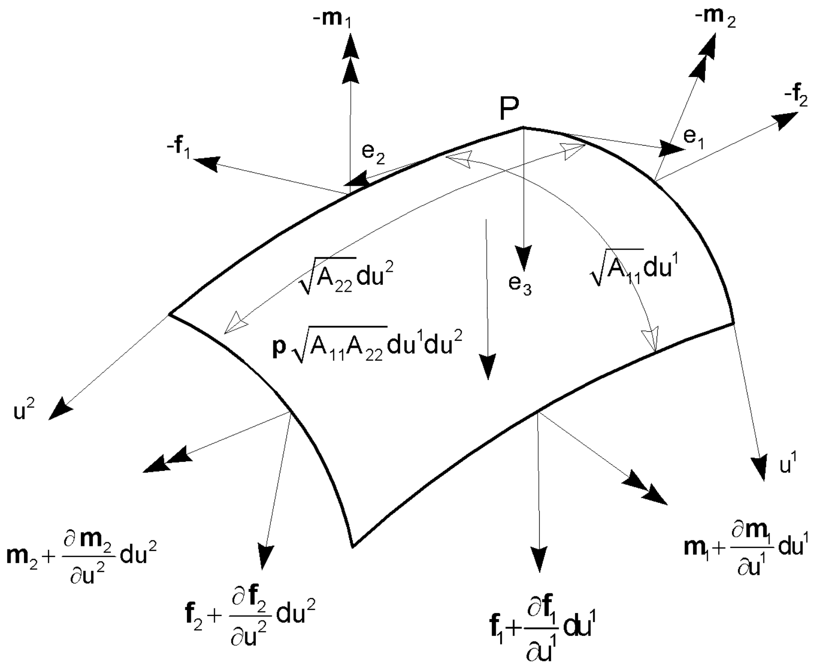
In figure 2, loads applied on shell element are shown: the unit force $\mathbf{p}\sqrt{A_{11}A_{22}}du^1du^2$ and the internal forces $\mathbf{f}_1$, $\mathbf{f}_2$ and couples $\mathbf{m}_1$ $\mathbf{m}_2$.

$$\mathbf{f}_1 = \left(N_{11}\,\mathbf{e}_1 + N_{12}\,\mathbf{e}_2 + T_{13}\,\mathbf{e}_3\right)\sqrt{A_{22}}du^2$$

$$\mathbf{f}_2 = \left(N_{21}\,\mathbf{e}_1 + N_{22}\,\mathbf{e}_2 + T_{23}\,\mathbf{e}_3\right)\sqrt{A_{11}}du^1$$

$$\mathbf{m}_1 = \left(M_{11}\,\mathbf{e}_2 - M_{12}\,\mathbf{e}_1\right)\sqrt{A_{22}}du^2$$

$$\mathbf{m}_2 = \left(M_{22}\,\mathbf{e}_1 - M_{21}\,\mathbf{e}_2\right)\sqrt{A_{11}}du^1$$



Let's write the equilibrium equations :
Equilibrium of momentum :

$$-\mathbf{f}_1 + \mathbf{f}_1 + \mathbf{f}_{1,1}\,du^1 - \mathbf{f}_2 + \mathbf{f}_2 + \mathbf{f}_{2,2}\,du^2 + \mathbf{p}\sqrt{A_{11}A_{22}}du^1du^2 = \mathbf{0}$$

Equilibrium of momentum of momentum

$$\mathbf{m}_1 + \mathbf{m}_{1,1}\,du^1 - \mathbf{m}_1 + \mathbf{m}_2 + \mathbf{m}_{2,2}\,du^2 - \mathbf{m}_2 + \sqrt{A_{11}}du^1\left(\mathbf{e}_1 \times \mathbf{f}_1\right) + \sqrt{A_{22}}du^2\left(\mathbf{e}_2 \times \mathbf{f}_2\right) = \mathbf{0}$$

Projecting on base vectors **$\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$.**
The derivatives may be obtain*ed from Gauss formulas.E.g taking in account

$$\mathbf{x}_{,1} = \mathbf{e}_1\sqrt{A_{11}}$$

we obtain

$$\mathbf{e}_{1,1} = -\frac{1}{\sqrt{A_{22}}}\left(\sqrt{A_{11}}\right)_{,2}\mathbf{e}_2 + \frac{\sqrt{A_{11}}}{R_1}\mathbf{e}_3$$

$$\mathbf{e}_{1,2} = -\frac{1}{\sqrt{A_{11}}}\left(\sqrt{A_{22}}\right)_{,1}\mathbf{e}_2$$

$$\mathbf{e}_{2,1} = -\frac{1}{\sqrt{A_{22}}}\left(\sqrt{A_{11}}\right)_{,2}\mathbf{e}_1$$

$$\mathbf{e}_{2,2} = -\frac{1}{\sqrt{A_{11}}}\left(\sqrt{A_{22}}\right)_{,1}\mathbf{e}_1 + \frac{\sqrt{A_{22}}}{R_2}\mathbf{e}_3$$

The derivatives of $\mathbf{e}_3$ are obtained from the formulas of Weingarten:

$$\mathbf{e}_{3,1} = -\frac{\sqrt{A_{11}}}{R_1}\mathbf{e}_1$$

$$\mathbf{e}_{3,1} = -\frac{\sqrt{A_{22}}}{R_2}\mathbf{e}_2$$

We obtain the six equilibrium equations

$$\left(N_{11}\sqrt{A_{22}}\right)_{,1} + \left(N_{21}\sqrt{A_{11}}\right)_{,2} + N_{12}\sqrt{A_{11}}_{,2} - N_{22}\sqrt{A_{22}}_{,1} - \frac{T_{13}\sqrt{A_{11}}\sqrt{A_{22}}}{R_1} + p_1\sqrt{A_{11}}\sqrt{A_{22}} = 0$$

$$\left(N_{22}\sqrt{A_{11}}\right)_{,2} + \left(N_{12}\sqrt{A_{22}}\right)_{,1} + N_{21}\sqrt{A_{22}}_{,1} - N_{11}\sqrt{A_{11}}_{,2} - \frac{T_{23}\sqrt{A_{11}}\sqrt{A_{22}}}{R_2} + p_2\sqrt{A_{11}}\sqrt{A_{22}} = 0$$

$$\left(T_{12}\sqrt{A_{22}}\right)_{,1} + \left(T_{23}\sqrt{A_{11}}\right)_{,1} + \frac{N_{11}\sqrt{A_{11}}\sqrt{A_{22}}}{R_1} + \frac{N_{22}\sqrt{A_{11}}\sqrt{A_{22}}}{R_2} + p_3\sqrt{A_{11}}\sqrt{A_{22}} = 0$$

$$\left(M_{12}\sqrt{A_{22}}\right)_{,1} + \left(M_{22}\sqrt{A_{11}}\right)_{,2} - M_{11}\sqrt{A_{11}}_{,2} + M_{21}\sqrt{A_{22}}_{,1} - T_{23}\sqrt{A_{11}}\sqrt{A_{22}} = 0$$

$$\left(M_{21}\sqrt{A_{11}}\right)_{,2} + \left(M_{11}\sqrt{A_{22}}\right)_{,1} - M_{22}\sqrt{A_{22}}_{,1} + M_{12}\sqrt{A_{11}}_{,2} - T_{13}\sqrt{A_{11}}\sqrt{A_{22}} = 0$$

$$N_{12} - N_{21} - \frac{M_{12}}{R_1} + \frac{M_{21}}{R_2} = 0$$

Novozhilov suggests to reduce the unknowns with the position:

$$M_{12} = M_{21}$$

$$S = N_{12} + \frac{M_{12}}{R_2} = N_{21} + \frac{M_{21}}{R_1}$$

The sixth equation is then satisfied and the remaining unknowns are: N_{11}, N_{22}, M_{12}, S, M_{11}, M_{22}, T_1, T_2.
You can find from the first two equations of momentum of momentum and substitute in the momentum T_{13} e T_{23}, obtaining three equations in the unknowns N_{11}, N_{22}, M_{12}, S, M_{11}, M_{22},

Membrane theory

Under special load and restraint conditions, moments and shears are small and may be disregarded . Then:

$$M_{11} = M_{22} = M_{12} = M_{21} = T_{13} = T_{23} = 0$$

The remaining equations, different from zero, are the following:

$$\left(N_{11}\sqrt{A_{22}}\right)_{,1} + \frac{1}{\sqrt{A_{11}}}\left(A_{11}N_{12}\right)_{,2} - N_{22}\sqrt{A_{22}}_{,1} + \sqrt{A_{11}}\sqrt{A_{22}}p_1 = 0$$

$$\left(N_{22}\sqrt{A_{11}}\right)_{,2} + \frac{1}{\sqrt{A_{22}}}\left(A_{22}N_{21}\right)_{,1} - N_{11}\sqrt{A_{11}}_{,2} + \sqrt{A_{11}}\sqrt{A_{22}}p_2 = 0$$

$$N_{11}\frac{1}{R_1}+N_{22}\frac{1}{R_2}+p_3=0$$

and the remaining unknowns are N_{11} ,N_{22} ,N_{12} .
This equilibrium is known as membrane equilibrium and the unknowns are statically determinate.
For instance in the spherical dome section of figure l, constraints rise reactions tangent to the base, then the reaction force N_{11} is symmetric and continuous if external loads are symmetric and continuous, the membrane equilibrium is then possible.
In figure 2 the base reactions are not tangent to the segmental dome. The membrane equilibrium is impossible without the chain AB.
This apparatus was made in Florence, in the Cathedral dome, it is made up by a chain in wood , a technical innovation for a dome not experienced by Romans. The chain was made by 24 wooden beams, three for each side of the octagon, connected together at the corners889 and along each side8169 of the octagon by iron brackets and bolts, surrounds the great dome near the button ,its utility has long been discussed.
. Della Porta, the construction manager, after the death of Michelangelo, of Saint Pietro, used iron chains and also Fontana suggested the technique.

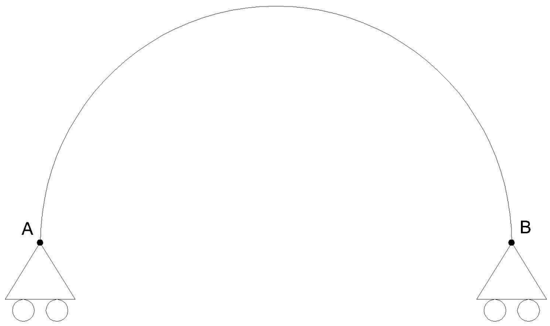


Figure 1 Membrane equilibrium of a dome. Constraints rightly posed.

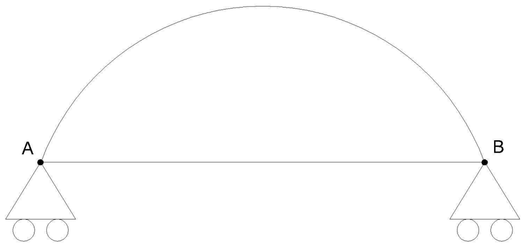


Figure 2 Membrane equilibrium of a segmental dome. External constraints are not correctly posed but the chain, 8supposed to be continous9, turns the reactions tangent to the dome base.

Revolution shells

Using spherical coordinates as natural :

$$u_1=\varphi \qquad u_2=\theta$$

e che

$$\sqrt{A_{\varphi\varphi}}=R_1 \qquad \sqrt{A_{\theta\theta}}=R_2\sin\varphi=r$$

$$\frac{dr}{d\varphi}=R_1\cos\varphi$$

Then from equations :

$$\left(N_{\varphi\varphi}r\right)_{,\varphi} + R_1 N_{\varphi\theta,\theta} - N_{\theta\theta}R_1\cos\varphi + rR_1 p_1 = 0$$
$$R_1 N_{\theta\theta,\theta} + \frac{1}{r}\left(r^2 N_{\varphi\theta}\right)_{,\varphi} + rR_1 p_2 = 0$$
$$N_{\varphi\varphi}\frac{1}{R_1} + N_{\theta\theta}\frac{1}{R_2} + p_3 = 0$$

<u>Revolution Shells symmetrically loaded</u>

Because of the symmetry

$$N_{\varphi\theta} = N_{\theta\varphi} = 0$$

and

$$p_2 = 0$$

From

$$\left(N_{\varphi\varphi}r\right)_{,\varphi} - N_{\theta\theta}R_1\cos\varphi + rR_1 p_1 = 0$$
$$N_{\varphi\varphi}\frac{1}{R_1} + N_{\theta\theta}\frac{1}{R_2} + p_3 = 0$$

..
:
The functions do not depend on θ.. Therefore, the second equilibrium equation is identically satisfied.
Replacing the 3rd equilibrium equation in the 1st, is an equation in Nφφ; multiplying by $\sin\varphi$:

$$\left(N_{\varphi\varphi}r\sin\varphi\right)_{,\varphi} + rR_1\left(p_1\sin\varphi + p_3\cos\varphi\right) = 0$$

Integrando da φ_0 a φ, supponendo anche che, per generalità, sul parallelo φ_0 sia presente un carico distribuito di valore q, si ha:
Integrate from , φ_0 to φ, and assume that, for generality, is present on the parallel a distributed load q, we have:

$$N_{\varphi\varphi}r\sin\varphi = -\int_{\varphi_0}^{\varphi} rR_1\left(p_1\sin\varphi + p_3\cos\varphi\right)d\varphi + qr_0$$

Moltiplicando l'equazione per 2π, l'equazione rappresenta l'equilibrio alla traslazione secondo la verticale tra le forze esterne applicate e le forze interne $N_{\varphi\varphi}$.
Determinato $N_{\varphi\varphi}$, si trova $N_{\theta\theta}$ con la terza equazione di equilibrio.
Multiplying the equation for 2π the equation is the equilibrium along the vertical between the externally applied forces and internal forces Nφφ.
You find Nφφ, and then Nθθ with the third equation of equilibrium.

Superficie sferica

$$p_3 = p\cos\varphi$$
$$p_1 = p\sin\varphi$$

$$N_{\varphi\varphi} = -\frac{1}{\sin^2\varphi}\int_{\varphi_0}^{\varphi} pR\sin\varphi d\varphi + qr_0$$

Supponiamo che la superficie sia una semisfera. Perciò $\varphi_0 = 0$

$$N_{\varphi\varphi} = -\frac{pr\left(1-\cos\varphi\right)}{\left(1-\cos^2\varphi\right)} = -\frac{pr}{\left(1+\cos\varphi\right)}$$

$$N_{\theta\theta} = pr\left(\frac{1}{1+\cos\varphi} - \cos\varphi\right)$$

Sulla chiave si ha:

$$N_{\varphi\varphi} = N_{\theta\theta} = -\frac{pr}{2}$$

N_{11} è sempre negativo, mentre $N_{\theta\theta}$ diviene positivo per $\left(\frac{1}{1+\cos\varphi} - \cos\varphi\right) = 0$, cioè è per $\varphi = 52°$ circa.

Per $\varphi = \frac{\pi}{2}$,

$$N_{\theta\theta} = -N_{\varphi\varphi} = pr$$

Se la cupola è ribassata fino a 52°, $N_{\theta\theta}$ è sempre negativo.

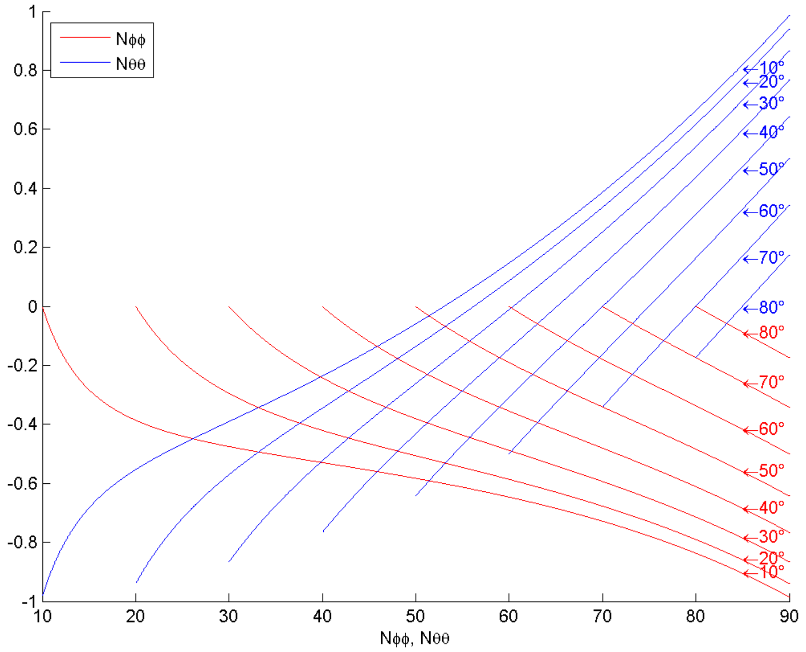
Caso della lanterna

Se la cupola sferica ha una lanterna, e q rappresenta il carico per unità di lunghezza da essa introdotto sulla cupola, si ha la soluzione:

$$N_{\varphi\varphi} = \frac{pR\left(\cos\varphi - \cos\varphi_0\right)}{\sin^2\varphi} - \frac{qr_0}{R\sin^2\varphi}$$

$$N_{\theta\theta} = -R\left(p\cos\varphi + \frac{N_{\varphi\varphi}}{R}\right)$$

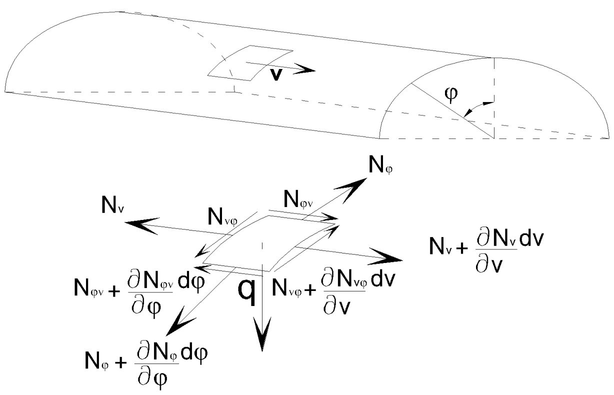
Costruzione di una cupola sferica



Il diagramma della figura è ricavato dalle funzioni precedenti adimensionalizzate ponendo q=0 e dividendo per pR. Se la costruzione è a cantilever procedendo dagli appoggi laterali verso il centro si può vedere dai diagrammi che l'equilibrio della cupola in costruzione sussiste ad iniziare da un angolo zenitale φ=80° e procedendo di 10° in 10° fino a φ=10°.

Superficie cilindrica

Già abbiamo definito la superficie cilindrica. Supponiamo in particolare che la retta **g** sia ortogonale al piano della curva C. Si ha allora, a seconda del tipo di curva, cilindri ellittici, circolari, parabolici e cosi via. Con riferimento alla Figura (xx), utilizzando coordinate cilindriche,



siano le coordinate gaussiane $u^1 = v$ e $u^2 = \varphi$. Risulta allora:

$$\sqrt{A_{vv}} = 1$$

e

$$\sqrt{A_{\varphi\varphi}} = R_2$$

$$R_1 \to \infty$$

Le equazioni del regime di membrana si specializzano nel caso presente nel modo seguente:

$$N_{vv,v} + \frac{1}{R_2} N_{v\varphi,\varphi} + p_1 = 0$$
$$N_{v\varphi,v} + \frac{1}{R_2} N_{\varphi\varphi,\varphi} + p_2 = 0$$
$$N_{\varphi\varphi} + R_2 p_3 = 0$$

Integrando la seconda equazione si ha:

$$N_{v\varphi} = -\left(p_2 + \frac{N_{\varphi\varphi,\varphi}}{R_2}\right)x + f_1(\varphi)$$

Sostituendo questa espressione nella prima e ponendo $p_1 = 0$, come nei casi pratici, si ha:

$$N_{vv} = \frac{1}{R_2}\left(\left(p_2 + \frac{N_{\varphi\varphi,\varphi}}{R_2}\right)_{,\varphi} \frac{x^2}{2} - f_{1,\varphi} x\right) + f_2(\varphi)$$

Supponiamo più precisamente che $R_2 = r$, cioè che il cilindro sia circolare. Si ha quindi:

$$p_2 = p\sin\varphi$$

e

$$p_3 = p\cos\varphi$$

Allora:

$$N_{\varphi\varphi} = -rp\cos\varphi$$

$$N_{v\varphi} = -2px\sin\varphi + f_1(\varphi)$$

$$N_{vv} = \frac{1}{r}\left(x^2 p\cos\varphi - f_{1,\varphi} x\right) + f_2(\varphi)$$

Determiniamo $f_1(\varphi)$ e $f_2(\varphi)$ con le condizione al contorno. Poiché la superficie è appoggiata per $x = 0$ e per $x = L$ su diaframmi rigidi nel loro piano, ma perfettamente flessibili fuori dal loro piano, risulterà $N_{vv} = 0$ per $x = 0$ e per $x = L$.Quindi:

$$f_2(\varphi) = 0$$
$$f_{1,2} = Lp\cos\varphi$$
$$f_1 = Lp\sin\varphi$$

Infine:

$$N_{vv} = \frac{x}{r}(x - L)p\sin\varphi$$

$$N_{\varphi\varphi} = -rp\cos\varphi$$
$$N_{v\varphi} = Lp\sin\varphi - 2px\sin\varphi$$

Si osserva che per $\varphi = \frac{\pi}{2}$, $N_{\varphi\varphi} = 0$, ma non così N_{12}:

$$N_{v\varphi} = Lp - 2px$$

che risulta positivo per $x < \frac{L}{2}$, negativo per $x > \frac{L}{2}$, nullo per $x = \frac{L}{2}$. È pertanto necessario lungo il bordo libero introdurre una trave.

Superficie conica

Equazioni di equilibrio

$$\left(N_{,vv}v\right)_v - N_{,\theta\theta} + vp_1 = 0$$
$$N_{,\theta\theta} + vp_3 tg\alpha = 0$$

Supponendo p1 e p_3 costante e sostituendo:

$$\left(N_{,vv}v\right)_v + vp_3 tg\alpha + vp_1 = 0$$

Integrando:

$$N_{,vv}v + \frac{v^2}{2}\left(p_3 tg\alpha + p_1\right) + cost = 0$$